

# Independence Complexes of Cylinders Constructed from Square and Hexagonal Grid Graphs

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**Abstract.** In the paper [Fendley et al., J. Phys. A: Math. Gen., 38 (2005), pp. 315–322], Fendley, Schoutens and van Eerten studied the hard square model at negative activity. They found analytical and numerical evidence that the eigenvalues of the transfer matrix with periodic boundary were all roots of unity. They also conjectured that for an  $m \times n$  square grid, with doubly periodic boundary, the partition function is equal to 1 when  $m$  and  $n$  are relatively prime. These conjectures were proven in [Jonsson, Electronic J. Combin., 13(1) (2006), R67]. There, it was also noted that the cylindrical case seemed to have interesting properties when the circumference of the cylinder is odd. In particular, when 3 is a divisor of both the circumference and the width of the cylinder minus 1, the partition function is -2. Otherwise, it is equal to 1. In this paper, we investigate the hard square and hard hexagon models at activity -1, with single periodic boundary, i.e., cylindrical identifications, using both topological and combinatorial techniques. We compute the homology groups of the associated independence complex for small sizes and suggest a matching which, we believe, with further analysis could help solve the conjecture. We also briefly review a technique recently described by Bousquet-Mélou, Linusson and Nevo, for determining some of the eigenvalues of the transfer matrix of the hard square model with cylindrical identification using a related, but more easily analysed model.

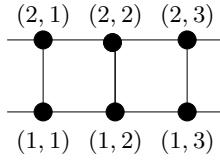
## 1 Introduction

Let  $G = (V, E)$  be an undirected graph. An *independent set*  $I$  in  $G$  is a subset of  $V(G)$  such that  $u, v \in V(G)$  implies  $(u, v) \notin E(G)$ . Now, take  $G$  to be the graph in Figure 1, where the half-edges are used to indicate that  $(1, 1)$  and  $(1, 3)$  are adjacent and that  $(2, 1)$  and  $(2, 3)$  are adjacent.

We are interested in the independent sets of  $G$  and start by listing them. We begin with the smallest set, the empty set  $\emptyset$ , which of course is an independent set in any graph. Then, the singleton sets  $\{(1, 1)\}, \{(1, 2)\}, \dots, \{(2, 3)\}$  are all independent. Finally, for each of the vertices  $(1, 1), (1, 2)$  and  $(1, 3)$  we can choose

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**Fig. 1.** An undirected graph  $G$ .

from two vertices of  $(2, 1)$ ,  $(2, 2)$  and  $(2, 3)$  to form a total of 6 independent sets of cardinality 2. We note that the alternating sum over all independent sets  $\sigma$ ,

$$\sum_{\sigma} (-1)^{|\sigma|} = 1. \quad (1)$$

It turns out that this happens a lot for graphs similar to  $G$ . Let the vertex set of  $C_{m,n}$  be  $(i, j)$  for  $1 \leq i \leq m$  and  $1 \leq j \leq n$  and let there be an edge between  $(i_1, j_1)$  and  $(i_2, j_2)$  if  $i_1 = i_2$  and  $|j_1 - j_2| = 1$  or if  $|i_1 - i_2| = 1$  and  $j_1 = j_2$ . Finally, let there be edges between each pair of vertices  $(i, 1)$  and  $(i, n)$ . With this construction, we have  $G = C_{2,3}$ . It is conjectured [9] that when  $m$  is odd and 3 is not a divisor of  $\gcd(m-1, n)$ , then the sum (1) over all independent sets  $\sigma$  of  $C_{m,n}$  equals 1 and that if  $m$  is odd and 3 does divide  $\gcd(m-1, n)$ , then this sum equals -2.

A *hard particle model* in statistical mechanics is defined on an underlying graph. A configuration of the model is a set of particles placed on the vertices of this graph such that no two particles are adjacent. One imagines that the particles have a certain size, or shape, that extends to their neighbouring vertices in the graph. Furthermore, the particles may not overlap, as they are considered *hard*. On a square grid, one talks about the hard square model, because of the shape of the particles. On a hexagonal grid, the particles are triangles as each vertex have three neighbours. The correspondence between configurations of hard particle models and independent sets on the underlying graph is clear.

The remarkable properties of the hard particle model on square grid graphs with cylindrical identifications were first observed by Fendley, Schoutens and van Eerten in [4]. They also studied the hard square model with doubly periodic boundary. Such identifications have the advantage that they make the graph vertex-transitive, i.e., the graph *looks* the same from every vertex. In this way the influence from the boundary is removed and these graphs are in that sense closer to the infinite square grid than the graphs obtained by simply taking an  $m$  by  $n$  subgraph of this grid. Fendley et al. conjectured that for the toroidal identifications, the roots of the characteristic polynomial of the associated transfer matrix were all roots of unity. Furthermore, they conjectured that when  $m$  and  $n$  were taken to be relatively prime, the alternating sum (1) vanished. These conjectures were proven in 2006 by Jonsson [9] in a purely combinatorial way.

To determine the partition function of these models at activity -1 has a direct correspondence in determining the Euler characteristic of the independence complex of the underlying graph. The matchings used in [9] were however not suitable for further topological analysis of the independence complex. Bousquet-Mélou, Linusson and Nevo [3] studied a different region of the square grid which turned

out to be more easily handled using Morse matchings. They also determined the homotopy type of the independence complex on a parallelogram in the square grid and related these findings to certain eigenvalues of the transfer matrix for the hard square model on  $C_{m,n}$ .

In this paper, we investigate the independence complexes for the square and hexagonal grids with cylindrical identifications. We obtain some results for small sizes by explicitly calculating the homology groups using a computer software package called Polymake [10]. We also construct Morse matchings for some classes of complexes with small circumference. The results obtained are enough to make certain conjectures on the homology groups. In particular, for odd  $n$ , there seems to be a symmetry in the square grid case and we conjecture that for  $i \geq 0$  and  $j, k \geq 1$ , we have

$$H_i(I(C_{j,2k+1})) \cong H_i(I(C_{k,2j+1})).$$

However, it is also clear that the homology does not behave nicely in many cases, even for odd  $n$ , which clearly reduces the applicability of this approach.

The paper is organised as follows. In Section 2 we present some basic notions and results from algebraic topology, discrete Morse theory and hard particle models. We use a particular type of Morse matching to determine the homology of some independence complexes related to the hard square and hard triangle models in Section 3. In Section 4, we introduce a different matching on the face poset of these independence complexes, which, while not being a Morse matching, still turns out to give some information on the alternating sum of the independence complexes of  $C_{m,n}$ . Finally, in Section 5 we review an idea from [3] which exploits the knowledge of the partition function of a hard square model on a different region in order to try and obtain parts of the spectrum of the transfer matrix for  $C_{m,n}$ .

## 2 Preliminaries

In this section we define simplicial complexes, and in particular independence complexes on grid graphs. We also review the basic results from discrete Morse theory and give a framework for constructing Morse matchings on independence complexes. Finally, we talk about the connection between hard particle models from statistical mechanics and independence complexes on grid graphs. For a detailed introduction to algebraic topology, simplicial complexes and homology, see for example [13] by Munkres. A classic reference for models in statistical mechanics is [2].

Throughout the paper we will use the following notation for addition and deletion of a single element  $x$  to or from a set  $S$ .

$$S + x := S \cup \{x\}, \quad S - x := S \setminus \{x\}.$$

To be precise about our base cases, we define the *Fibonacci numbers*  $F_i$  in the following way.

$$F_0 = 0, F_1 = 1, \text{ and } F_{i+2} = F_{i+1} + F_i \text{ for } i \geq 0.$$

## 2.1 Simplicial Complexes

Let  $X$  and  $Y$  be two topological spaces. If there is a continuous map  $f : X \rightarrow Y$ , with a continuous inverse, then  $X$  and  $Y$  are said to be *homeomorphic*, denoted by  $X \cong Y$ .

Two continuous maps  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are called *homotopic* if there is a continuous map  $F : X \times [0, 1] \rightarrow Y$  such that  $F(x, 0) = f(x)$  and  $F(x, 1) = g(x)$ . The spaces  $X$  and  $Y$  are said to be *homotopy equivalent*, denoted by  $X \simeq Y$ , if there are maps  $f : X \rightarrow Y$  and  $h : Y \rightarrow X$  such that  $f \circ h$  is homotopic to the identity map on  $Y$  and  $h \circ f$  is homotopic to the identity map on  $X$ . A space  $X$  that is homotopy equivalent to a point is called *contractible*. This will be denoted by  $X \simeq \bullet$ .

An (*abstract*) *simplicial complex*  $\Delta$ , is a collection of subsets over a ground set  $V$  such that if  $\sigma \in \Delta$  and  $\tau \subseteq \sigma$ , then  $\tau \in \Delta$ . The elements of  $\Delta$  are called *faces*, and the maximal faces are called *facets*. The *dimension* of a face  $\sigma$  is defined to be  $\dim(\sigma) = |\sigma| - 1$ .

The *reduced Euler characteristic*  $\tilde{\chi}$  of a simplicial complex  $\Delta$  is the following sum.

$$\tilde{\chi}(\Delta) := \sum_{\sigma \in \Delta} (-1)^{\dim(\sigma)} \quad (2)$$

Given a simplicial complex  $\Delta$ , the *homology group*  $H_i(\Delta)$  of  $\Delta$  in dimension  $i$ , over a ring  $R$ , is defined by  $H_i(\Delta) = C_i(\Delta)/B_i(\Delta)$ . Here, the groups  $C_i$  and  $B_i$  are the groups of  $i$ -chains and  $i$ -boundaries, respectively. The rank of  $H_i(\Delta)$  is called the  *$i$ th Betti number* and is denoted by  $\beta_i$ . The following relation to the Euler characteristic, which can alternatively be taken as its definition, is called the *Euler-Poincaré formula*.

$$\tilde{\chi}(\Delta) = \sum_{i \geq -1} (-1)^i \beta_i \quad (3)$$

Here,  $\beta_{-1}$  is defined to be 1 if the complex  $\Delta = \{\emptyset\}$  and 0 otherwise. Homotopy equivalent spaces have isomorphic homology groups so  $\tilde{\chi}$  is a topological invariant.

For a topological space  $X$ , we let  $\partial X$  denote the *boundary* of  $X$  and  $\text{int } X = X \setminus \partial X$  the *interior* of  $X$ . The  $n$ -dimensional simplex  $\Delta_n$  is the full simplicial complex on a  $n + 1$ -point ground set, i.e.,  $\Delta_n = 2^V$ , with  $|V| = n + 1$ . The  $n$ -dimensional *ball*  $B^n$  is the subspace of the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  given by  $B^n := \{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\| \leq 1\}$ . The  $n$ -dimensional *sphere*  $S^n$  is similarly given by  $S^n := \{\mathbf{v} \in \mathbb{R}^{n+1} \mid \|\mathbf{v}\| = 1\}$ . That is,  $S^n = \partial B^{n+1}$ . Apparently, we have  $B^n \cong \Delta_n$  and  $S^n \cong \partial \Delta_{n+1}$ , where the simplicial complex  $\partial \Delta_{n+1}$  can be described by  $2^V \setminus V$ , for some  $|V| = n + 1$ .

Let  $\Delta$  be a simplicial complex. The *cone* over  $\Delta$ , denoted by  $\text{cone}(\Delta)$ , is the simplicial complex over the ground set of  $\Delta$  together with a new vertex  $c$ , the cone point.

$$\text{cone}(\Delta) = \{\sigma, \sigma + c \mid \sigma \in \Delta\}$$

The cone over any simplicial complex is contractible.

The *suspension* over  $\Delta$ , denoted by  $\text{susp}(\Delta)$ , is the following simplicial complex over the ground set of  $\Delta$  together with two new vertices  $s_0$  and  $s_1$ .

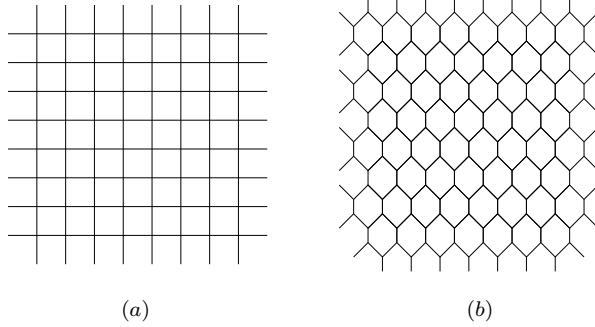
$$\text{susp}(\Delta) = \{\sigma, \sigma + s_0, \sigma + s_1 \mid \sigma \in \Delta\}$$

For any simplicial complex  $\Delta$  and  $i \geq 0$ ,

$$H_i(\Delta) \cong H_{i+1}(\text{susp}(\Delta)). \quad (4)$$

## 2.2 Independence Complexes of Regular Grids

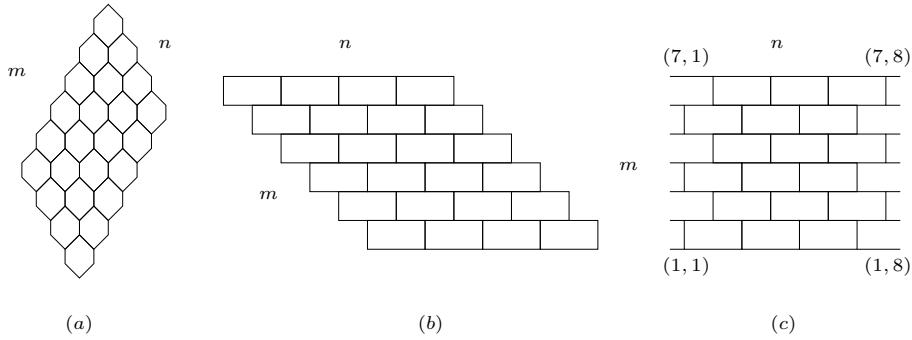
Regular grids are tilings of the 2-dimensional plane by regular convex polygons. There exist three types of regular tilings, triangular, square and hexagonal. In this paper we have considered the square and the hexagonal grid. These are depicted in Figure 2.



**Fig. 2.** (a) The square grid. (b) The hexagonal grid.

Let  $S$  be the graph induced by the boundary of the square grid. Then,  $S$  is isomorphic to an infinite graph on the vertex set  $\mathbb{Z} \times \mathbb{Z}$ , with edges between  $(u_1, u_2)$  and  $(v_1, v_2)$  if  $u_1 = v_1$  and  $|u_2 - v_2| = 1$  or if  $|u_1 - v_1| = 1$  and  $u_2 = v_2$ . From now on, we will assume that  $S$  is defined in this way. Let  $S_{m,n}$  be the graph induced from  $S$  by the vertex set  $[m] \times [n]$ , where  $[m]$  denotes the set  $\{1, \dots, m\}$ . By identifying sides of  $S_{m,n}$ , we can obtain graphs embeddable on different 2-dimensional topological surfaces. Specifically, let  $C_{m,n}$  be  $S_{m,n+1}$ , with vertices  $(i, n+1)$  and  $(i, 1)$  identified, for  $i \in [m]$ . Furthermore, let  $T_{m,n}$  as the graph  $S_{m+1,n+1}$ , with the vertices  $(i, n+1)$  and  $(i, 1)$  as well as the vertices  $(m+1, j)$  and  $(1, j)$  identified, for  $i \in [m]$  and  $j \in [n]$ , respectively. The graph  $C_{m,n}$  is obviously embeddable on a cylinder (but not on the plane), while  $T_{m,n}$  is embeddable on a torus (but not on a cylinder).

Let  $H$  be the graph induced by the boundary of the hexagonal grid. Similarly to the case of  $S$ , we can find an appropriate representation of the graph  $H$  with vertex set  $\mathbb{Z} \times \mathbb{Z}$ . This graph has edges between  $(u_1, u_2)$  and  $(v_1, v_2)$  if  $u_1 = v_1$  and  $|u_2 - v_2| = 1$  or if  $u_1 - v_1 = 1$ ,  $u_2 = v_2$ , and  $u_1 + u_2 \equiv_2 0$ . For positive integers  $m$  and  $n$  let  $H_{m,n}$  be the subgraph of  $H$  shown in Figure 3(a). We can straighten this graph and draw it on the square grid, as is shown in (b). We will use this *brick wall*-like appearance in this paper. If we identify the left and right boundaries of the graph in Figure 3(b), we get a graph  $C_{m,n}^H$  with the topology of a cylinder. Figure 3(c) shows an alternative way of drawing this cylinder, using half edges on the left and right boundaries to indicate the identifications. We will



**Fig. 3.** (a)  $H_{6,4}$  as a subgraph of  $H$ . (b)  $H_{6,4}$  as a brick wall. (c) The cylinder  $C_{6,4}^T$ .

use the vertex set  $[m] \times [n] \subseteq \mathbb{Z} \times \mathbb{Z}$  to represent these graphs, with the vertices numbered as in Figure 3(c). We will also briefly look at the hexagonal grid with toroidal identifications. In this case, we will use the representation in Figure 3(b), identifying the left and right boundaries as well as the top and bottom ones. We denote the graph obtained by these identifications by  $C_{m,n}^T$ . Note that this graph is different from the one we would obtain by doing the top to bottom identifications in Figure 3(c).

We will be studying simplicial complexes related to these regular grids. Let  $G = (V, E)$  be a finite, simple and undirected graph. An *independent set*  $I$  in  $G$  is a subset of the vertex set  $V(G)$  such that if  $u, v \in I$  then  $(u, v) \notin E(G)$ . That is, no pair of neighbours may be in  $I$  at the same time.

Define the *independence complex*  $I(G)$  as the simplicial complex on the ground set  $V(G)$  with  $\sigma \in I(G)$  if and only if  $\sigma$  is an independent set in  $G$ . Similarly, let  $X(G)$  be the *flag complex* of  $G$ . This is the simplicial complex on the ground set  $V(G)$  with  $\sigma \in X(G)$  if and only if  $\sigma$  induces a complete subgraph in  $G$ . It is obvious from the definitions that  $I(G) = X(\bar{G})$ , where  $\bar{G} = (V, \bar{E})$  denotes the complement graph of  $G$  with respect to the complete graph on  $V$ , i.e.,  $\bar{E} = \binom{V}{2} \setminus E$ .

In general, computing  $I(G)$  is believed to be hard, since the facets of  $I(G)$  are the maximal independent sets of  $G$ . Any such facet of maximal dimension therefore gives a solution to the *maximum independent set problem*, which is a well known NP-complete problem [11]. Dual to this is the NP-complete *maximum clique problem* of finding a complete subgraph (or *clique*) in  $G$  of maximal size. These cliques are given by the highest dimensional facets of  $X(G)$ .

### 2.3 Discrete Morse Theory

*Discrete Morse theory* was introduced in [5] by Robin Forman. It relates the homotopy type of a simplicial complex to that of a *CW-complex* with (ideally) simpler structure. See [6] for a hands-on introduction to this subject.

Let  $\Delta$  be a simplicial complex. We define the *face poset*  $P(\Delta)$  on  $\Delta$  as the set of faces in  $\Delta$  ordered by inclusion. The *underlying graph* of the Hasse diagram of  $P(\Delta)$ , or simply the underlying graph of  $P(\Delta)$ , is the directed graph with the

faces of  $\Delta$  as vertices and the cover relations of  $P(\Delta)$  as edges, directed from the smaller faces to the larger faces. Let  $M$  be a matching on the underlying graph of  $P(\Delta)$ . We will also say that  $M$  is a matching on the complex  $\Delta$ . The *graph induced by  $M$*  is the graph obtained from the underlying graph of the Hasse diagram by reversing the direction of the edges which are in  $M$ .

**Definition 2.1.** A Morse matching  $M$  on a simplicial complex  $\Delta$  is a (partial) matching of the Hasse diagram of  $P(\Delta)$  such that the graph induced by  $M$  is acyclic.

The main theorem of Morse theory relates the homotopy type of a simplicial complex to that of a certain *CW-complex* related to the unmatched faces of a Morse matching. These unmatched faces are sometimes referred to as the *critical cells* of the matching. For our purposes, the following definition of a finite CW-complex will suffice.

**Definition 2.2.** A finite CW-complex (or cell complex)  $C$  is a collection of cells such that for each cell  $e_\alpha$

- (a) There is a homeomorphism from  $B^k$  to  $e_\alpha$  for some  $k$ .
- (b)  $e_\alpha \setminus \text{int } e_\alpha$  is the union of cells  $e_\beta$  in  $C$ .

The main theorem of discrete Morse theory can be stated as follows, using the notion of a Morse matching.

**Theorem 2.1.** Let  $\Delta$  be a simplicial complex with a Morse matching  $M$ . Assume that for each  $p \geq 0$ , there are  $u_p$  unmatched  $p$ -simplices. Then  $\Delta$  is homotopy equivalent to a CW-complex with exactly  $u_p$  cells of each dimension  $p$ , for each  $p > 0$ , and  $u_0 + 1$  cells of dimension 0.

**Corollary 2.2** Let  $\Delta$  be a simplicial complex with a Morse matching  $M$  such that  $u_p = 0$  for all but one  $p$ . Then, for this particular  $p$ ,

$$\Delta \simeq \bigvee_{u_p} S^p.$$

That is,  $\Delta$  is homotopy equivalent to a wedge of  $u_p$   $p$ -dimensional spheres.

**Corollary 2.3** Let  $\Delta$  be a simplicial complex with a perfect Morse matching  $M$ . Then,  $\Delta$  is contractible.

In this paper, we will apply discrete Morse theory to independence complexes. The acyclic matchings that we will use are all instances of the following general construction. By describing them in this framework, we can easily determine their acyclicity. The construction is similar to the *matching trees* of [3]. Throughout this paper, we will however use the term *matching tree* to refer to our construction.

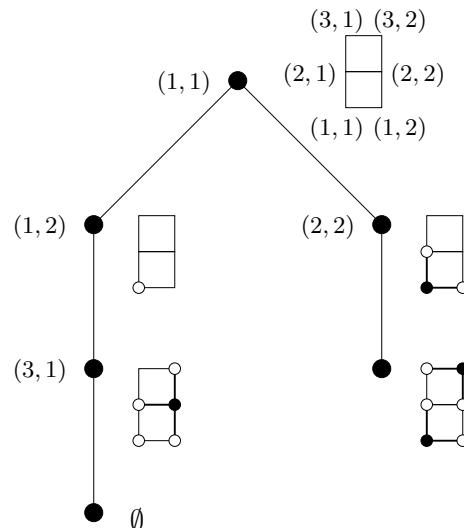
*Example 2.1.* The following example shows the basic idea behind the matching trees. In Figure 4, we construct a matching on the complex  $I(S_{3,2})$ . First we partition the set of faces in to two disjoint sets, based on whether the vertex  $(1, 1)$  is in the face or not. We indicate this by writing  $(1, 1)$  to the left of the root in the tree. To the right of the root we place the graph  $S_{3,2}$  which indicates that

all faces of the complex are still unmatched. In the left subtree we have forced  $(1, 1)$  not to appear in any face. We illustrate this by placing an empty particle on  $(1, 1)$  in the graph to the right of the root of this subtree. In the right subtree we have instead forced  $(1, 1)$  to be present in every face. This is illustrated by a filled particle on  $(1, 1)$  and empty particles on its neighbours  $(1, 2)$  and  $(2, 1)$ . In both subtrees, this effectively reduces the problem to the construction of a matching on two smaller graphs, induced from  $S_{3,2}$  by the complement of the fixed vertices. In the next step, the vertex  $(1, 2)$  in the left graph, and  $(2, 2)$  in the right graph both have degree 1. We use these vertices to create the matchings  $M(1, 2)$  and  $M(2, 2)$  in the respective subtree. Let  $\Sigma_l = I(S_{3,2} - (1, 1))$  and  $\Sigma_r = I(S_{3,2} \setminus \{(1, 1), (1, 2), (2, 1)\})$ , and define the matchings as follows.

$$M(1, 2) := \{(\sigma, \sigma - (1, 2)) \mid (1, 2) \in \sigma \in \Sigma_l\},$$

$$M(2, 2) := \{(\sigma, \sigma - (2, 2)) \mid (2, 2) \in \sigma \in \Sigma_r\}.$$

The unmatched faces  $\sigma$  remaining after removing the faces matched by  $M(1, 2)$  must have  $(1, 2), (2, 1), (3, 2) \notin \sigma$  and  $(2, 2) \in \sigma$ . This further reduces the graph. In the right subtree, it completely fixes one single graph. In the left subtree, there is still a vertex  $(3, 1)$  on which we can match. Matching  $\{(2, 2)\}$  with  $\{(2, 2), (3, 1)\}$  removes all the faces from this subtree, which we indicate by writing  $\emptyset$  to the right of the last node.



**Fig. 4.** A matching tree for the simplicial complex  $I(S_{3,2})$ .

In Proposition 2.6, we will see that the union of the matchings made in this type of construction forms an acyclic matching on the original complex. We can therefore apply Theorem 2.1, and due to the remaining 1-dimensional face in the right subtree, which is the only critical cell, we conclude that  $I(S_{3,2})$  is homotopy equivalent to a 1-dimensional sphere.

**Definition 2.3 (Matching tree).** Let  $T$  be a finite, rooted tree in which each internal node has either one or two children. To each internal node (which includes the root)  $t \in T$ , we associate a vertex  $p(t) \in G$ .

We say that  $T$  is a matching tree if the set of vertices  $p(t)$  fulfil the following two conditions. For any node  $t_r \in T$ , let  $(t_1, \dots, t_r)$  be the path from the root  $t_1$  to  $t_r$ . Then,

- all  $p(t_1), \dots, p(t_r)$  are distinct, and
- $\{p(t) \mid t \in \{t_1, \dots, t_r\} \cap V_m\}$  forms an independent set in  $G$ ,

where  $V_m$  denotes the set of nodes with one child.

The vertices  $p(t)$  are called the *pivots* in [3]. They are the vertices that we wrote to the left of the nodes in Figure 4. Given the complex  $I(G)$  and the tree  $T$ , we determine a matching on  $P(I(G))$  as follows. To each node  $t \in T$ , we associate a *content*  $C(t)$ , which is a subset of the complex  $I(G)$ . This is the set that was illustrated by the graphs to the right of the nodes in Figure 4. For the root  $r \in T$ , we let  $C(r) = I(G)$ , and for the other nodes the content will be determined recursively.

If  $t$  has two children, then we call  $t$  a *split node*, and we let the content of the root of the left subtree of  $t$  be the set of faces  $\{\sigma \in C(t) \mid p(t) \notin \sigma\}$  and the content of the root of the right subtree of  $t$  be the faces  $\{\sigma \in C(t) \mid p(t) \in \sigma\}$ .

If  $t$  has only one child, then we call  $t$  a *match node*. We define the partial matching  $M(t)$  associated to the match node and the set of matched faces  $F(t)$  as follows.

$$M(t) := \{(\sigma, \sigma - p(t)) \mid p(t) \in \sigma \in C(t)\}, \quad F(t) := \bigcup_{(\sigma, \tau) \in M(t)} \{\sigma, \tau\}$$

We will see below that the set  $M(t)$  forms a partial matching on  $C(t)$ . For match nodes, we let the content of the child of  $t$  be  $C(t) \setminus F(t)$ .

**Definition 2.4.** Let  $G$  be a graph and  $\sigma \in I(G)$ . We say that a position  $x \in V(G)$  is free in  $\sigma$  if  $x \notin \sigma$  and  $\sigma + x$  is an independent set in  $G$ .

The following lemma provides all the structure we need on the sets  $C(t)$  to show acyclicity for our matchings.

**Lemma 2.4** Let  $T$  be a matching tree and let  $t \in V_m$ . If  $\sigma \in C(t)$  and  $p(t)$  is free in  $\sigma$ , then  $\sigma + p(t) \in C(t)$ . Conversely, if  $\sigma \in C(t)$  and  $p(t) \in \sigma$ , then  $\sigma - p(t) \in C(t)$ .

*Proof.* Both parts of the lemma obviously hold true for the root node. We will show that if either part fails for some  $t$ , where  $t$  is taken to be the first node on the path from the root to  $t$  for which this is the case, then the other part must fail at an earlier node  $t'$ , closer to the root. If we apply this argument twice, we reach a node  $t''$  which is strictly between the root and  $t$  and for which the same part of the lemma fails. This contradicts our choice of  $t$ , and the lemma follows.

Assume that  $\sigma \in C(t)$  and that  $p(t)$  is free in  $\sigma$ . The reason for  $\sigma + p(t)$  not to be in  $C(t)$  could be that there was a split node which had  $\sigma$  in one subtree

and  $\sigma + p(t)$  in the other. This can not happen, however, since the two faces only differ in  $p(t)$  and along the path from the root, all pivots must be distinct.

The only remaining possibility is then that  $\sigma + p(t)$  was matched in an earlier node  $t'$ . Now, if  $p(t')$  is free in  $\sigma + p(t)$ , then it is also free in  $\sigma$ , and by our assumption,  $\sigma$  would have been matched at  $t'$  as well. So, we can assume that  $p(t') \in \sigma + p(t)$ , and then also  $p(t') \in \sigma$ . The second part of the lemma therefore fails at  $t'$ .

For the second part, assume that  $\sigma \in C(t)$  and  $p(t) \in \sigma$ , but that  $\sigma - p(t) \notin C(t)$ . Again,  $\sigma - p(t)$  could not have been removed by an earlier split node since it only differs in  $p(t)$  from  $\sigma$ .

So assume that  $\sigma - p(t)$  was matched in an earlier node  $t'$ . If  $p(t') \in \sigma - p(t)$ , then  $p(t') \in \sigma$  and by induction,  $\sigma$  would have been matched with  $\sigma - p(t')$  at  $t'$ . Therefore,  $p(t')$  is free in  $\sigma - p(t)$ . By the restriction that the pivots of the match nodes form an independent set in  $G$ , we have that  $p(t')$  is free in  $\sigma$  as well. The first part of the lemma therefore fails at  $t'$ .  $\square$

Using Lemma 2.4, it is easy to see that in fact, for  $t \in V_m$ , we have

$$F(t) \subseteq C(t).$$

That is, for each match node  $t \in V_m$ ,  $M(t)$  is a partial matching on the subgraph of the underlying graph of  $P(I(G))$  induced by the set  $C(t)$ . It is acyclic, as the only way to “move down” in the directed graph induced by  $M(t)$  is by removing  $p(t)$ .

It therefore makes sense to define the matching  $M$  determined by the tree  $T$  as

$$M := \bigcup_{t \in V_m} M(t). \quad (5)$$

We will now show that  $M$  is acyclic, so that Theorem 2.1 is applicable. The critical cells will be those in the set  $\bigcup_{t \in V_l} C(t)$ , where  $V_l$  denotes the set of leaves in  $T$ . Our proof will use the following lemma, which appears in [8].

**Lemma 2.5 (Cluster Lemma)** *Let  $\Delta$  be a simplicial complex and  $f : P(\Delta) \rightarrow Q$  be a poset map, where  $Q$  is an arbitrary poset. For  $q \in Q$ , let  $M_q$  be an acyclic matching on  $f^{-1}(q)$ . Let*

$$M = \bigcup_{q \in Q} M_q.$$

*Then,  $M$  is an acyclic matching on  $\Delta$ .*

The following proposition justifies our definition of a matching tree, showing that any tree gives an acyclic matching on the corresponding independence complex. The proof is entirely guided by the Cluster lemma (Lemma 2.5).

**Proposition 2.6** *Let  $G$  be a graph and  $T$  a matching tree defined on the independence complex  $I(G)$ . Then, the matching  $M$  given by (5) on  $I(G)$  is acyclic.*

*Proof.* We start by defining a total order on the nodes of the tree  $T$ . First, if  $t'$  is an ancestor of  $t$ , then let  $t \leq_T t'$ . Then, if  $T_1$  and  $T_2$  are two subtrees of a node  $t$ , with  $T_1$  being the subtree where  $p(t) \notin \sigma$  for any  $\sigma \in C(t_1)$ , let  $t_1 \leq_T t_2$  for all  $t_1 \in T_1$  and  $t_2 \in T_2$ .

Let  $Q = (V(T), \leq_T)$  and

$$f^{-1}(t) = \begin{cases} F(t) & \text{if } t \in V_m, \\ C(t) & \text{if } t \in V_l, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to check that the  $F(t)$  are pairwise disjoint and that  $F(t_1) \cap C(t_2) = \emptyset$  for all  $t_1 \in V_m$  and  $t_2 \in V_l$ . Furthermore, each face of  $I(G)$  is either in  $F(t_1)$  for some  $t_1 \in V_m$  or in  $C(t_2)$  for some  $t_2 \in V_l$ . Thus,  $f : I(G) \rightarrow Q$  is a uniquely defined map. It remains to verify that it is indeed a poset map.

Let  $\sigma, \tau \in I(G)$ , with  $\sigma \subseteq \tau$ . If  $f(\sigma)$  and  $f(\tau)$  are in different subtrees of some node  $t$ , then  $\sigma$  must be in the left, as we know that  $p(t)$  is in all faces of the right subtree, but in none of the left. Thus,  $f(\sigma) \leq_T f(\tau)$ .

If instead  $f(\sigma)$  and  $f(\tau)$  are on the same path from the root to some leaf, then we will assume that  $f(\tau)$  is closer to the root than  $f(\sigma)$ . Let  $v = p(f(\tau))$  be the pivot at  $f(\tau)$ . If  $v \in \sigma$ , then Lemma 2.4 says that  $\sigma$  should have been matched in the node  $f(\tau)$ , so we have a contradiction. Therefore,  $v \notin \sigma$ . But then  $\sigma \subseteq \tau - v$ , and  $\tau - v$  was matched with  $\tau$ . This implies that  $v$  is free in  $\sigma$ , and Lemma 2.4 again shows that  $\sigma$  should have been matched in  $f(\tau)$ . We have a contradiction. Therefore  $f(\tau)$  must be equal to or lie further from the root than  $f(\sigma)$ , which means that  $f(\sigma) \leq_T f(\tau)$ .

Finally, for each  $t \in V_m$ , we have that  $M(t)$  is a perfect, acyclic matching of  $f^{-1}(t) = F(t)$ . We can therefore use Lemma 2.5 to conclude that  $M$  is acyclic on the entire  $I(G)$ .  $\square$

We finally note that our construction disallows some strategies for choosing split and match nodes which would have resulted in acyclic matchings. This is due to the constraint that the match nodes in each path from the root must form an independent set. This condition is not strictly necessary, but for the purposes of this article, this construction will suffice.

## 2.4 Hard particle models

A particular type of model in two-dimensional statistical mechanics is the *hard particle model*. In such a model, given a regular grid like the ones described in Section 2.2, a configuration is a set of particles placed on the sites of the grid so that no two particles are adjacent. The name is explained by replacing each particle with a polygon, using the neighbours of the original particle as the vertices of the polygon. This polygon is assumed to be “hard”, meaning that no two polygons may overlap in a configuration. When the grid is the square grid, the polygons are squares, and the model referred to as the *hard square model*. For the hexagonal grid, the polygons are triangles, and one has the *hard triangle model*. For a thorough introduction to models in statistical mechanics, see [2], in which can also be found the solution to the hard particle model on a triangular grid. This solution was first presented by Baxter in [1] in 1980.

To a model, one can associate a *partition function*  $Z$ . This is the generating function of the configurations with respect to the number of particles of a configuration.

$$Z = \sum_{\sigma} z^{|\sigma|}, \tag{6}$$

where the sum is taken over all allowed configurations of the model. The variable  $z$  of  $Z$  is called the *activity* and controls the behaviour of the model.

A commonly used framework for deriving the partition function is the *transfer matrix method*. See [14, Chapter 4.7] for an introduction to this method. Here, we illustrate the idea by giving a transfer matrix for the rectangle  $S_{m,n}$ . More examples are given in Section 5.

Let  $L_n$  denote the set of independent sets on the line  $S_{1,n}$ . Since  $|L_1| = 2$ ,  $|L_2| = 3$ , and the recursive relation  $|L_{n+2}| = |L_{n+1}| + |L_n|$  holds, we have  $|L_n| = F_{n+2}$ , the Fibonacci numbers. Let  $T(z) := T(z)_{\sigma\tau}$  be the  $F_{n+2} \times F_{n+2}$  matrix where the individual matrix elements are given by

$$T(z)_{\sigma\tau} = \begin{cases} z^{|\tau|} & \text{if } \sigma \cap \tau = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

The matrix  $T(z)$  is called a transfer matrix of the model. It describes the effect of adding an additional row. We will only be studying models at activity -1 so we leave out the variable  $z$  and write  $T = T(-1)$ . The generating function of  $Z(S_{m,n})$  with respect to  $n$  can now be written as

$$\sum_{n \geq 0} Z(S_{m,n})t^n = (1 - tT)^{-1}.$$

In addition, by simply taking the trace of the  $n$ th power of the matrix  $T$ , we can also express the partition function  $Z(C_{m,n})$  of a cylinder.

$$Z(C_{m,n}) = \text{tr}(T(z)^n),$$

or as a generating function

$$\sum_{n \geq 0} Z(C_{m,n})t^n = \text{tr}(1 - tT)^{-1}.$$

The *characteristic polynomial*  $P(t)$  of  $T$  is given by

$$P(t) = \det(tI - T).$$

The roots of the characteristic polynomial can also be found by taking the poles of the generating function  $\text{tr}(1 - tT)^{-1}$ .

It was observed by Fendley, Schoutens and van Eerten [4] that the hard square model (on a torus) exhibit some interesting properties when the activity is chosen to be  $-1$ . Among other things, they conjectured that all the roots of the characteristic polynomial are roots of unity. These conjectures were rigorously proven by Jonsson [9] by means of applying an ingeniously designed series of matchings on the set of configurations. Doing this, Jonsson discovered an underlying structure of a particular type of rhombus tilings of the plane which he used to settle the conjectures. An attempt to mimic the idea of Jonsson is carried out in Section 4, albeit with limited results.

The following conjecture for the hard particle model on a cylinder in the square grid at activity  $-1$  can be found in [9]. The conjecture is based on results for  $m \leq 11$ .

*Conjecture 2.1.* For odd  $n$ ,

$$Z(C_{m,n}) = \begin{cases} -2 & \text{if } 3 \mid \gcd(m-1, n), \\ 1 & \text{otherwise.} \end{cases}$$

From (2) and (6) we see that, at  $z = -1$ ,

$$Z(G) = -\tilde{\chi}(I(G)).$$

That is, the partition function at activity  $-1$  is (up to a negative sign) equal to the reduced Euler characteristic of the independence complex of the underlying graph. For this reason, we will sometimes refer to  $Z(G)$  as the *alternating sum* of the corresponding independence complex. In the next section, we investigate if this connection to topology can be taken further than just to the reduced Euler characteristic.

### 3 Explicit results

In this section we present the explicitly calculated homology groups of some independence complexes on graphs related to the square and the hexagonal grid. First, we do some small cases on the square grid by hand, for a fixed circumference  $n$ . Then, we present some larger cases, where we have used a computer to calculate the homology groups. The calculations were made using the software package Polymake [10] version 2.2 by Michael Joswig and Ewgenij Gawrilow with many contributors.

#### 3.1 Results for small circumference

For some cases with small  $n$ , we can find suitable matching trees and work out the homotopy type. We do this here for the square grid when  $n \leq 5$  and for the hexagonal grid when  $n = 1$  or  $n = 2$ . For the case  $n = 4$ , we are only able to determine the alternating sum. This result is given in Proposition 3.7. We note that for the square grid, these results are consistent with Conjecture 2.1.

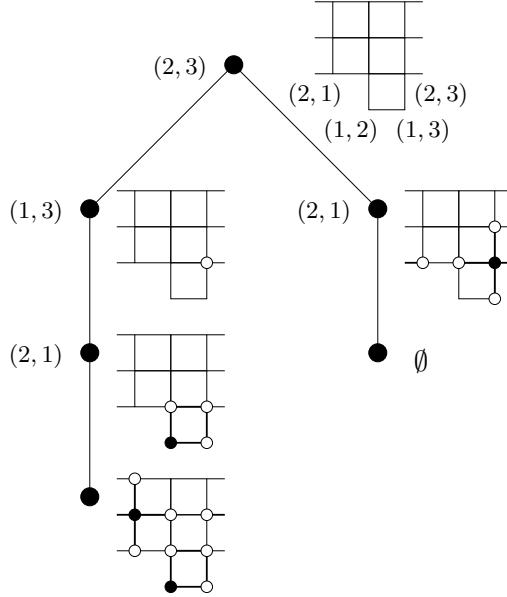
When  $n = 1$  there can be no particles on the grid since every vertex in the grid is a neighbour to itself. Therefore  $I(C_{m,1}) \cong \{\emptyset\}$  and  $Z(C_{m,1}) = 1$ .

**Proposition 3.1** *Let  $m \geq 1$ . Then,  $I(C_{m,2}) \simeq S^{\lceil m/2 \rceil - 1}$  and  $Z(C_{m,2}) = (-1)^{\lceil m/2 \rceil - 1}$ .*

*Proof.* Note that  $C_{m,2} = S_{m,2}$ . The construction in Example 2.1 in Section 2 is easily extended to arbitrary  $m$ . The only unmatched configuration has  $m/2$  particles when  $m$  is even and  $(m+1)/2$  particles when  $m$  is odd.  $\square$

**Proposition 3.2** *Let  $m \geq 1$ . Then,*

$$I(C_{m,3}) \simeq \begin{cases} S^{2m/3-1} & \text{if } m \equiv_3 0, \\ \bigvee_2 S^{2(m-1)/3} & \text{if } m \equiv_3 1, \\ S^{2(m-2)/3+1} & \text{if } m \equiv_3 2, \end{cases}$$



**Fig. 5.** Matching tree for  $C_{m,3} - (1,1)$ .

and

$$Z(C_{m,3}) = \begin{cases} 1 & \text{if } m \equiv_3 0, \\ -2 & \text{if } m \equiv_3 1, \\ 1 & \text{if } m \equiv_3 2. \end{cases}$$

*Proof.* Let  $C'_{m,n} = C_{m,n} - (1,1)$ . We construct an acyclic matching on  $C_{m,3}$  as follows. If we start by splitting on  $(1,1)$ , we are left with  $C'_{m,3}$  in the left subtree, and after a matching, a graph isomorphic to  $C'_{m-1,3}$  in the right subtree. Figure 3.1 shows the matching tree construction for  $C'_{m,3}$ . Note that the last remaining graph in the left subtree is isomorphic to  $C'_{m-3,3}$  and that we have added two particles on the path from the root to this node.

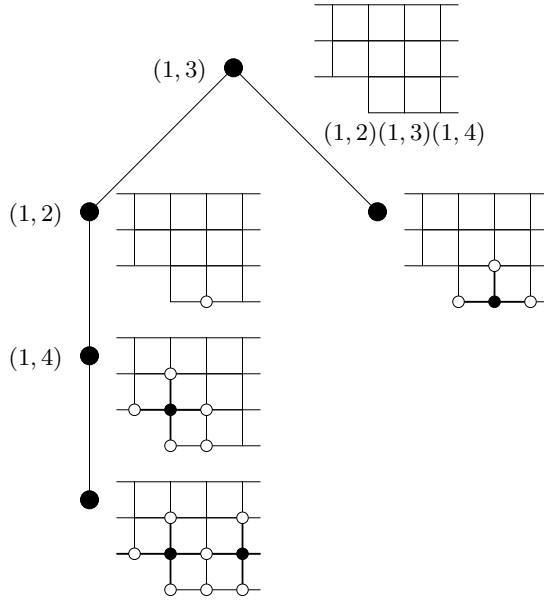
When  $m = 1$ , we can construct a matching leaving two configurations with one particle each, when  $m = 2$ , we find a perfect matching, and when  $m = 3$ , we are left with one configuration, having two particles. The proposition follows by adding the configurations from the two subtrees, and using Proposition 2.6 and Corollary 2.2.  $\square$

**Proposition 3.3** *Let  $m \geq 1$ . Then,*

$$I(C_{m,4}) \simeq \begin{cases} \bigvee_{m+1} S^{m-1} & \text{if } m \text{ is even,} \\ \bigvee_m S^{m-1} & \text{if } m \text{ is odd,} \end{cases}$$

and

$$Z(C_{m,4}) = \begin{cases} m+1 & \text{if } m \text{ is even,} \\ -m & \text{if } m \text{ is odd.} \end{cases}$$



**Fig. 6.** Matching tree for  $C_{m,4} - (1,1)$ .

*Proof.* Let  $C'_{m,n} = C_{m,n} - (1,1)$ . Figure 3.1 shows how to construct a matching tree for the graph  $C'_{m,4}$ . Note that the left subtree gives a recursive relation of length 2. When  $m$  is even, there will be one configuration with  $m$  particles left at the end of this subtree. When  $m$  is odd, we have a perfect matching. In the right subtree, there remains a graph which is isomorphic to  $C'_{m-1,n}$ . When  $m=1$ , this subtree has one configuration with one particle. Thus, the entire tree leaves  $\lfloor m/2 \rfloor + 1$  configurations with  $m$  particles.

Now, we build the matching tree for  $C_{m,4}$  as follows. We first split on  $(1,1)$ . The left subtree will be the one previously constructed for  $C'_{m,4}$  with  $\lfloor m/2 \rfloor + 1$  remaining configurations. In the right subtree, we can match on  $(3,1)$ , leaving us with a graph isomorphic to  $C'_{m-2,4}$ , and  $\lfloor (m-2)/2 \rfloor + 1$  remaining configurations.

We then use Proposition 2.6 to show that we have an acyclic matching with  $2\lfloor m/2 \rfloor + 1$  unmatched configurations, each with  $m$  particles. Corollary 2.2 finally gives us the homotopy type.  $\square$

**Proposition 3.4** *Let  $m \geq 1$ . Then,*

$$I(C_{m,5}) \simeq \begin{cases} S^{m-1} & \text{if } m \text{ is even,} \\ S^m & \text{if } m \text{ is odd,} \end{cases}$$

and

$$Z(C_{m,5}) = 1.$$

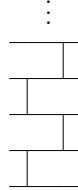
*Proof.* We give a sketch of the matching tree for  $C_{m,5}$ . First, we split on both  $u = (1,1)$  and  $v = (3,1)$ . For  $S \subseteq 2^{\{u,v\}}$ , let

$$X_m(S) = \{\sigma \in I(C_{m,5}) \mid \sigma \cap \{u, v\} = S\}.$$

Then,  $X_m(\{u\})$  and  $X_m(\{v\})$  can be perfectly matched. If  $m$  is even, then after matching  $X_m(\emptyset)$ , one configuration with  $m$  particles remain, and there is a perfect matching for  $X_m(\emptyset)$  otherwise. For  $X_m(\{u, v\})$ , we have one configuration with  $m + 1$  particles left if  $m$  is odd and a perfect matching otherwise.  $\square$

**Proposition 3.5** *Let  $m \geq 1$ . Then,*

$$I(C_{m,1}^H) \simeq \begin{cases} S^{2m/3+1} & \text{if } m \equiv_3 0 \\ \bullet & \text{if } m \equiv_3 1 \\ S^{2(m+1)/3} & \text{if } m \equiv_3 2 \end{cases}$$



**Fig. 7.** The graph  $C_{4,1}^H$  is isomorphic to  $P_9$ .

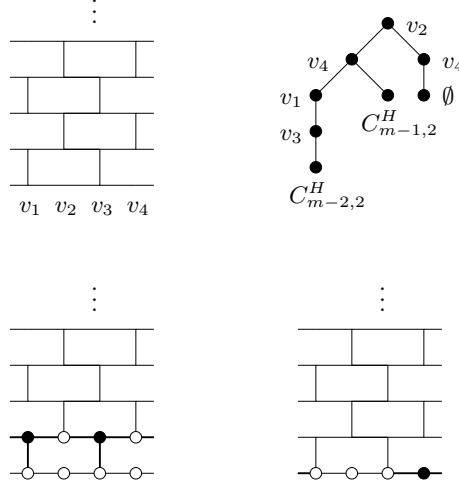
*Proof.* It can be seen in Figure 7 that the graph  $C_{m,1}^H$  is isomorphic to  $P_{2n+1}$ , the path on  $2n + 2$  vertices. The result immediately follows.  $\square$

**Proposition 3.6** *Let  $m \geq 1$ . Then, the simplicial complex  $I(C_{m,2}^H)$  is homotopy equivalent to a wedge of  $F_{m+2}$   $m$ -dimensional spheres.*

*Proof.* The Morse matching for  $I(C_{m,2}^H)$  is given by the matching tree in Figure 8. We start by splitting on vertex  $v_2$ . If there is a particle on  $v_2$ , then we can find a perfect matching on all remaining configurations by matching on  $v_4$ . If  $v_2$  is empty, we split on vertex  $v_4$ . The remaining graph in the right subtree is shown in the lower right corner of the figure. We see that it is isomorphic to  $C_{m-1,2}^H$ , and that we have added one particle on the way to this node. In the left subtree, we eventually reach a node where the remaining graph is the one in the lower left corner. This graph is isomorphic to  $C_{m-2,2}^H$  and the two match nodes on the way to this node adds two particles.

The base cases when  $m = 1$  and  $m = 2$  are constructed by hand. We can construct a matching on  $I(C_{1,2}^H)$  so that two configurations remain, both with two particles. On  $I(C_{2,2}^H)$  we can similarly construct a matching which leaves three configurations, each with three particles. Using these matchings and Proposition 2.6, we finally construct an acyclic matching on  $I(C_{m,2}^H)$  which leaves  $F_{m+2}$  unmatched configurations each with  $m + 1$  particles.  $\square$

As we will see in the next section when we determine some homology groups using computer calculations, the complex  $I(C_{5,4}^H)$  has non-zero homology in two dimensions. Therefore, we can not in general hope to apply Corollary 2.2 to the



**Fig. 8.** A recursion for  $I(C_{m,2}^H)$ .

complexes  $I(C_{m,4}^H)$ . However, Table 5 suggests that the alternating sum may be simple. We are able to find a matching tree which shows that this alternating sum is identical to that of  $I(C_{m,2}^H)$  which was determined in Proposition 3.6. This results is given in the following proposition.

**Proposition 3.7** *Let  $m \geq 1$ . Then,  $Z(C_{m,4}^H) = (-1)^{m+1} F_{m+2}$ .*

*Proof.* We give a sketch of the matching tree construction used to prove Proposition 3.7. We start by splitting on the vertices  $v_1, \dots, v_4$  in Figure 9. The first thing to note is that if particles are present on two consecutive  $v_i$  and  $v_{i+1}$  (modulo 4), then we can match everything in this subtree. The upper right configuration shows this case, when for all faces  $\sigma$  in this subtree,  $v_1, v_2 \in \sigma$ . Here, we match on  $v_5$ . A similar situation occurs when only one  $v_i$  is present.

Therefore, we have either the case  $v_1, v_3 \in \sigma, v_2, v_4 \notin \sigma$  (or the complement), or no  $v_i$  is in  $\sigma$ . The lower left configuration shows the first case, where we proceed by matching on  $v_5$  and  $v_6$ . In this case, which occurs twice, we end up with the graph  $C_{m-2,4}^H$  and 4 particles fixed in each configuration.

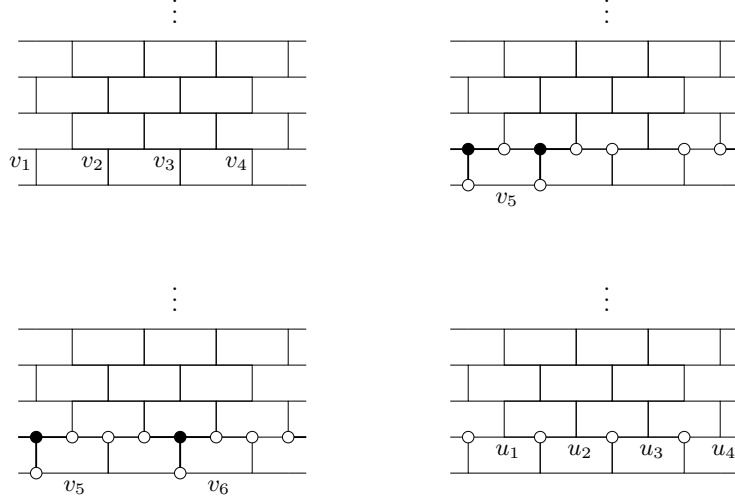
The lower right configuration shows the case when none of the  $v_i$  are in  $\sigma$ . We then match on  $u_1, \dots, u_4$ . In the second case, we reach the graph  $C_{m-3,4}^H$  after fixing 7 particles. We see here that the critical cells will be spread out over several dimensions. We therefore focus on the alternating sum, disregarding the homotopy type in this case. From the previous argument, we have the recursion

$$Z(C_{m,4}^H) = 2Z(C_{m-2,4}^H) - Z(C_{m-3,4}^H),$$

and base cases are

$$Z(C_{0,4}^H) = -1, Z(C_{1,4}^H) = 2, Z(C_{2,4}^H) = -3,$$

where  $C_{0,4}^H$  is isomorphic to an 8-cycle. The proposition follows by solving this recursion.  $\square$



**Fig. 9.** Split and match nodes in a matching tree for  $I(C_{m,4})$ .

### 3.2 The matching $M_O$

In the rest of this section we will present the homology groups of some independence complexes which have been computer generated. To reduce the size of the complexes and the time of the calculations, we first apply a particular Morse matching to each complex. We start by introducing this matching, and the manipulations made on the complex, before giving our results in Section 3.3 and Section 3.4

Let  $G$  be a simple, undirected graph. Let  $O \in I(G)$  be an independent set in  $G$ , which we will call the *odd vertices*. We will assume a fixed, but arbitrary total order  $<$  on  $O$ .

The matching  $M_O$  on the Hasse diagram of  $P(I(G))$  is defined as follows. The edge  $(\sigma, \sigma')$  is in  $M_O$  if  $\sigma' \setminus \sigma = \{o\}$ , where  $o \in O$  is the smallest element (with respect to  $<$ ) in  $O$  for which  $\sigma \cup \{o\}$  is an independent set. According to Proposition 2.6, the matching  $M_O$  is acyclic, and therefore a Morse matching. According to Theorem 2.1, we then know that  $I(G)$  is homotopy equivalent to some cell-complex, with cells in the dimensions given by the sizes of the unmatched faces in  $I(G)$ . We call this set of unmatched faces  $X$  and note that we can express it as

$$X = \{\sigma \in I(G) \mid \forall u \in O : \sigma \cap Nbd(u) \neq \emptyset\}, \quad (7)$$

where  $Nbd(u)$  denotes the neighbourhood of  $u$  in  $G$ . Of course, in general,  $X$  does not form a simplicial complex. From (7) we see that the total order  $<$  does not affect  $X$ . Now, given a set  $O$ , and thereby a set  $X$ , we define two simplicial complexes  $\Delta_O$  and  $\Gamma_O$ .

$$\Delta_O = \{\sigma \mid \sigma \subseteq \sigma' \in X\}, \quad \Gamma_O = \Delta_O \setminus X$$

We will now prove a theorem that relates the homotopy type of the original independence complex to that of  $\Gamma_O$ . First, however, we will give a topological

lemma. In [12] it was stated for the case when  $\Delta$  is a simplex. This slightly more general version is needed for the proof of our theorem.

**Lemma 3.8** *Let  $\Gamma \subseteq \Delta$  be simplicial complexes and assume that  $\Delta$  is contractible. Then,  $\Delta/\Gamma \simeq \text{susp}(\Gamma)$ .*

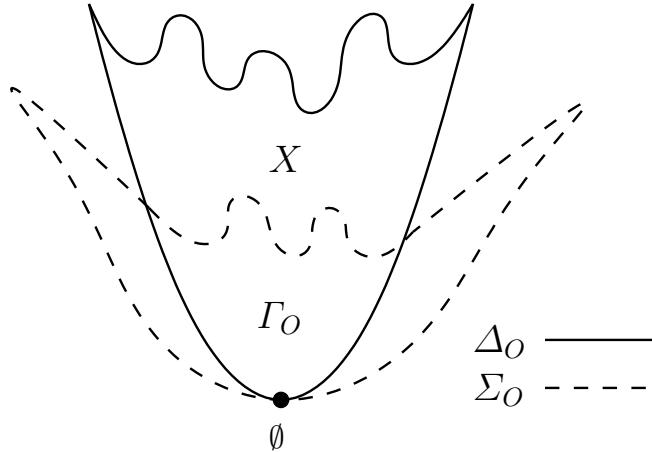
*Proof.* The concept of a *mapping cone* is described in [7, Chapter 0]. In our case, the mapping cone is the simplicial complex  $X = \Delta \sqcup_{\text{id}} \text{cone}(\Gamma)$ , i.e., we raise a cone in  $\Delta$  over the subcomplex  $\Gamma$ . Note that  $X/\text{cone}(\Gamma) \simeq \Delta/\Gamma$  and that  $X/\Delta \simeq \text{susp}(\Gamma)$ . Now, since both  $\text{cone}(\Gamma)$  and  $\Delta$  are contractible, we have  $X/\text{cone}(\Gamma) \simeq X \simeq X/\Delta$  which finishes the proof.  $\square$

We are now ready to state and prove the following theorem, which allows us to simplify our calculations of the homology groups.

**Theorem 3.9.** *Let  $O \in I(G)$  be an independent set and assume that  $\Delta_O$  is contractible. Then,*

$$I(G) \simeq \text{susp}(\Gamma_O). \quad (8)$$

*Proof.* The set  $X$  is given by (7). Another way to express this is to say that  $\sigma \in X$  if and only if  $O \subseteq \bigcup_{u \in \sigma} Nbd(u)$ . From the latter definition we can immediately see that if  $\sigma \subseteq \sigma'$  and  $\sigma \in X$ , then we have  $\sigma' \in X$ . Let  $\Sigma_O = I(G) \setminus X$ . The previous argument shows that  $\Sigma_O$  is a simplicial complex. Figure 10 illustrates the relationship between  $X$  and the simplicial complexes  $\Delta_O, \Gamma_O$  and  $\Sigma_O$ .



**Fig. 10.** The set  $X$  and the simplicial complexes  $\Delta_O, \Gamma_O$  and  $\Sigma_O$

Since  $M_O$  gives a perfect matching on  $\Sigma_O$ , we can collapse  $\Sigma_O$  to a point in  $I(G)$ . But this is the same as identifying  $\Gamma_O$  with a point in  $\Delta_O$ . Thus,

$$I(G) \simeq I(G)/\Sigma_O \simeq \Delta_O/\Gamma_O.$$

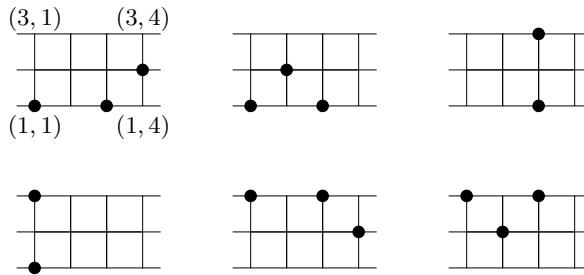
Using Lemma 3.8, we now see that  $I(G) \simeq \text{susp}(\Gamma_O)$ .  $\square$

The reason for using  $\Gamma_O$  instead of  $I(G)$  is two-fold. First, we have used a matching to reduce the size of the complex, and thereby, hopefully, reduced the time to compute its homology. Secondly, in many cases a description of the complex  $\Gamma_O$  is easier to obtain and more compact than a description of the original complex.

### 3.3 Graphs on the square grid

Let  $G = C_{m,n}$  and assume that  $n$  is even. In this case we choose the independent set  $O = \{u = (i,j) \in V(G) \mid i+j \text{ is odd}\}$ . Then,  $\Delta_O \cong \Delta_{mn/2-1}$  and thus contractible. The complex  $\Gamma_O$  can be generated from the set  $O$  in the following way. For each  $u \in O$  we have a maximal face  $\{v \in V(G) \setminus O \mid v \notin Nbd(u)\} \in \Gamma_O$ . We see that for the maximal faces  $\sigma \in \Gamma_O$ , either  $\dim(\sigma) = mn/2 - 5$  or  $\dim(\sigma) = mn/2 - 4$ , depending on whether the corresponding  $v \in V(G) \setminus O$  is on the (non-identified) boundary of the cylinder or not.

*Example 3.1.* Let  $m = 3$ ,  $n = 4$ , and  $O = \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4)\}$ . Then, the six maximal faces of  $\Gamma_O$  are as shown in Figure 11 and the complex itself can be realised as in Figure 12. We can see that it is homotopy equivalent to a wedge of three 1-dimensional spheres. Therefore, we conclude from Theorem 3.9 that  $I(C_{3,4}) \simeq \bigvee_3 S^2$ .



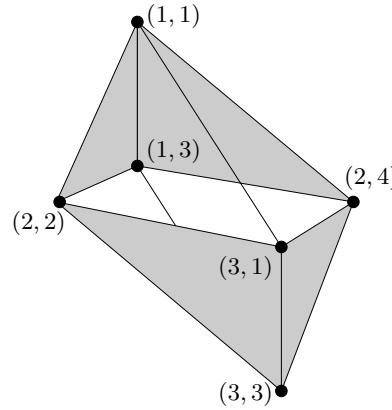
**Fig. 11.** The six maximal faces of  $\Gamma_O$ .

For odd  $n$ , we let  $O = \{u = (i,j) \in V(G) \mid j \neq n, i+j \text{ is odd}\}$ . The complex  $\Delta_O$  is still contractible but the maximal faces of  $\Gamma_O$  are less regular.

Table 1 and Table 2 show the calculated homology groups for the complex  $I(C_{m,n})$  for small  $m$  and  $n$ . An entry  $(k,d)$  in the table should be read as  $H_k(I(C_{m,n})) \cong \mathbb{Z}^d$ . Missing entries are cases where Polymake ran for too long, or more often, ran out of memory. The homology vanishes in all dimensions that are not listed. Note that for  $n \leq 5$ , the homology groups in these tables can be derived, for arbitrary  $m$ , from the results in Section 3.1. For odd  $n$ , we observe an interesting symmetry, which we pose as a conjecture.

*Conjecture 3.1.* For all  $i \geq 0$  and  $j, k \geq 1$ , the following holds.

$$H_i(I(C_{j,2k+1})) \cong H_i(I(C_{k,2j+1}))$$



**Fig. 12.** The simplicial complex  $\Gamma_O \simeq \bigvee_3 S^1$ .

Table 3 is taken from [9] and shows the values of  $Z(C_{m,n})$  for small  $m$  and  $n$ , calculated using transfer matrices. The cases when  $n$  is odd are left out as they all satisfy Conjecture 2.1. In these cases, it was also observed, and conjectured to hold in general, that all the roots of the characteristic polynomials were roots of unity. In particular, this implies that the rows of Table 3 are periodic.

		m							
		1	2	3	4	5	6	7	8
n	2	(0,1)	(0,1)	(1,1)	(1,1)	(2,1)	(2,1)	(3,1)	(3,1)
	4	(0,1)	(1,3)	(2,3)	(3,5)	(4,5)	(5,7)	(6,7)	(7,9)
	6	(1,2)	(2,1)	(3,1)	(5,4)	(6,1)	(7,1)	(8,2)	(9,4)
	8	(2,1)	(3,3)	(5,5)	(7,5)	(8,1), (9,4)	(11,7)		
	10	(2,1)	(4,1)	(7,1)	(8,1), (9,2)				
	12	(3,2)	(5,3)	(8,3)	(11, 8)				
	14	(4,1)	(6,1)	(9,1)					

**Table 1.** Homology of  $I(C_{m,n})$  for even  $n$ .

Both for even and odd  $n$ , we find cases where there is non-zero homology in more than one dimension. This means that in these cases there is no easy way to determine the alternating sums via Morse matchings and the homotopy type. In the odd  $n$ -case, the *bad* example shows up for the graph  $C_{4,9}$ , where  $Z(C_{4,9}) = -2$ . It is therefore still possible that the independence complexes in the other odd  $n$ -cases, where  $Z(C_{m,n})$  is conjectured to be 1, are simple.

### 3.4 Graphs on the hexagonal grid

Take any finite subgraph  $H'$  of the hexagonal grid  $H$ . As in the case of the square grid with even  $n$ , there is a natural notion of *even* and *odd* vertices given by the

		m										
		1	2	3	4	5	6	7	8	9	10	11
n	3	(0,2)	(1,1)	(1,1)	(2,2)	(3,1)	(3,1)	(4,2)	(5,1)	(5,1)	(6,2)	(7,1)
	5	(1,1)	(1,1)	(3,1)	(3,1)	(5,1)	(5,1)	(7,1)	(7,1)	(9,1)	(9,1)	
	7	(1,1)	(3,1)	(5,1)	(5,1)	(7,1)	(9,1)	(11,1)				
	9	(2,2)	(3,1)	(5,1)	(7,1), (8,3)	(9,1)						
	11	(3,1)	(5,1)	(7,1)	(9,1)							
	13	(3,1)	(5,1)	(9,1)								
	15	(4,2)	(7,1)	(11,1)								
	17	(5,1)	(7,1)									
	19	(5,1)	(9,1)									

**Table 2.** Homology of  $I(C_{m,n})$  for odd  $n$ .

$Z(C_{m,n})$		m										
		1	2	3	4	5	6	7	8	9	10	11
n	2	-1	-1	1	1	-1	-1	1	1	-1	-1	1
	4	-1	3	-3	5	-5	7	-7	9	-9	11	-11
	6	2	-1	1	4	-1	-1	4	1	-1	2	1
	8	-1	3	5	5	3	7	1	1	-1	3	-3
	10	-1	-1	1	1	9	-1	1	1	-11	-1	1
	12	2	3	-3	8	-5	7	8	9	-9	14	-11
	14	-1	-1	1	1	-1	13	1	1	13	-1	15
	16	-1	3	5	5	3	7	1	33	-1	3	13
	18	2	-1	1	4	-1	-1	22	1	-1	38	1
	20	-1	3	-3	5	5	7	-7	9	41	11	-11
	22	-1	-1	1	1	-1	-1	1	23	-1	-1	89
	24	2	3	5	8	3	7	16	1	-1	78	-3

**Table 3.**  $Z(C_{m,n})$  for small  $m$  and  $n$ , when  $n$  is even. Taken from Jonsson [9].

parity of the vertices. Let  $G$  be the graph  $H'$ , possibly with some identifications made on vertices of the same parity and let  $O$  be the set of odd vertices in  $G$ , where the parity is inherited from  $H$ . Then,  $\Delta_O$  is a simplex of dimension  $|V(G)| - |O(G)| - 1$  and thus contractible. The complex  $\Gamma_O$  is determined in the same way as was done for the square grid with even  $n$ . For each  $u \in O$  we have a maximal face  $\{v \in V(G) \setminus O \mid v \notin Nbd(u)\} \in \Gamma_O$ .

Table 4 shows the homology of the complex  $I(C_{m,n}^H)$  for small  $m$  and  $n$ . As before, an entry  $(k, d)$  means that  $H_k(I(C_{m,n}^H)) \cong \mathbb{Z}^d$  and that the homology vanishes in all other dimensions. As noted in the square grid case, the entries for  $n = 2$  could be derived, for arbitrary  $m$  from Proposition 3.6. We note with interest that there seems to be a relation between the dimensions in which homology appears for  $C_{i,j}^H$  and  $C_{j-1,i+1}^H$ , but the data is quite limited. Table 5 shows some of the first values of the alternating sum  $Z(C_{m,n}^H)$ . They have been generated using transfer matrices. We have not found any nice factorisation of these polynomials and they seem to have mostly roots which are not roots of unity.

		m						
		2	3	4	5	6	7	8
n	2	(2,3)	(3,5)	(4,8)	(5,13)	(6,21)	(7,34)	(8,55)
	3	(4,5)	(6,7)	(7,3)	(9,22)	(11,23)	(12,24)	
	4	(6,3)	(7,4), (9,1)	(10,8)	(11,8), (13,5)			
	5	(7,6)	(10,11)					
	6	(9,15)	(11,4), (13,13)					
	7	(11,8)						
	8	(12,19)						

**Table 4.** Homology of  $I(C_{m,n}^H)$ .

$Z(C_{m,n}^H)$		m											
		1	2	3	4	5	6	7	8	9	10	11	12
n	1	0	1	-1	0	1	-1	0	1	-1	0	1	-1
	2	2	-3	5	-8	13	-21	34	-55	89	-144	233	-377
	3	0	-5	-7	3	22	23	-24	-92	-67	141	367	152
	4	2	-3	5	-8	13	-21	34	-55	89	-144	233	-377
	5	0	6	-11	-5	51	-76	-60	416	-536	-655	3351	-3646
	6	2	15	17	55	160	231	886	1664	3947	11121	21065	59296
	7	0	8	-15	-35	57	34	-42	687	20	-4207	-2379	3611
	8	2	-19	37	-88	533	-725	3466	-11927	21417	-105552	273881	-682665
	9	0	-41	-43	183	958	941	-9924	-22943	19265	289806	587437	-1949599

**Table 5.**  $Z(C_{m,n}^H)$  for small  $m$  and  $n$ .

In the case of the hexagonal lattice, we have also calculated the homology for the toroidal case, i.e., for the complex  $I(T_{m,n}^H)$ . Table 6 shows the homology for

small cases. Table 7 lists  $Z(C_{m,n}^T)$  for some small values of  $m$  and  $n$  calculated using the transfer matrices. For  $m \leq 4$ , the characteristic polynomials  $P(m)$  of these matrices factor reasonably well. After removing powers of  $t$ , the polynomials are given by

$$\begin{aligned} P(1) &= 1 + t + t^2 \\ P(2) &= 1 + t - t^2 \\ P(3) &= (1 - t + t^2)(1 - t + 2t^2 + t^3) \\ P(4) &= (1 + t - t^2)(1 - t)^3(1 + t)^3(1 + t^2) \end{aligned}$$

For larger values of  $m$ , the polynomials do not factor as nicely, and seem to have mostly roots which are not roots of unity.

		m						
		2	3	4	5	6	7	8
n	2	(1,3)	(2,4)	(3,7)	(4,11)	(5,18)	(6,29)	(7,47)
	3	(2,4)	(4,10)	(6,4)	(7,17)	(9,32)	(10,1), (11,3)	(12,76)
	4	(3,7)	(6,4)	(7,15)	(9,1), (10,12)	(11,20), (12,1), (13,3)		

**Table 6.** Homology of  $I(T_{m,n}^H)$ .

$Z(T_{m,n}^H)$		m						
		1	2	3	4	5	6	7
n	1	-1	-1	2	-1	-1	2	-1
	2	-1	3	-4	7	-11	18	-29
	3	2	-4	-10	-4	17	32	2
	4	-1	7	-4	15	-11	22	-29
	5	-1	-11	17	-11	-51	127	-36
	6	2	18	32	22	127	192	394
	7	-1	-29	2	-29	-36	394	552
	8	-1	47	-76	55	-411	1478	83

**Table 7.**  $Z(T_{m,n}^H)$  for small  $m$  and  $n$ .

## 4 A second matching

In this section we introduce an alternative matching on the face poset of the complex  $I(C_{m,n})$ . While in Section 3, the matching presented was always acyclic, we will make no such claims here. This means that we can not rely on the discrete Morse theory for our results. Still, we will be able to derive results on the alternating sum  $Z(C_{m,n})$ . A similar idea was used with great success on the hard square model on a torus in [9].

Let  $M$  be any matching on the underlying graph of  $P(I(C_{m,n}))$ . Then, for all  $(\sigma, \tau) \in M$ , we have  $(-1)^{|\sigma|} + (-1)^{|\tau|} = 0$  and therefore,

$$\sum_{\sigma \in I(C_{m,n})} (-1)^{|\sigma|} = \sum_{\sigma \in X} (-1)^{|\sigma|},$$

where we let  $X$  denote the set of unmatched faces in  $I(C_{m,n})$ . Thus, we can calculate  $Z(C_{m,n})$  using only the faces in the set  $X$ .

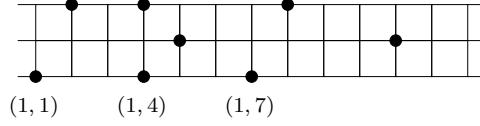
In order to describe our matching we need to introduce some notation and terminology. Let  $\sigma$  be a face in  $I(C_{m,n})$ . Define

$$\pi(\sigma) := \{(1, j) \in \sigma\}.$$

Let  $\pi(\sigma) = \{x_1, \dots, x_k\}$  be ordered by the second component of the  $x_i$ . For  $1 \leq i \leq k$ , we define the *interval*  $N_i$  of  $\sigma$  at  $x_i = (1, j_i)$  to be the set

$$N_i = \{(1, j_{i-1} + 1), (1, j_{i-1} + 2), \dots, (1, j_i)\},$$

where the addition is modulo  $n$  in the second component, and the indices 0 and  $k$  are identified. Let  $N(\sigma) = \{N_1, \dots, N_k\}$  be the set of all intervals. We call the sequence  $[|N_1|, \dots, |N_k|]$  the *signature* of  $\sigma$  and we identify signatures which are identical up to cyclic rotations. We distinguish between *even* and *odd intervals* by the parity of their sizes and between *even* and *odd positions* in an interval  $N_i$  by saying that the element  $x_i$  is in an even (odd) position if  $N_i$  is an even (odd) interval, and extending this notion of parity to the rest of the interval. Let  $\pi(\sigma) = \pi_o(\sigma) \cup \pi_e(\sigma)$ , where  $\pi_o$  is the set of particles in odd position, and  $\pi_e(\sigma)$  the set of particles in even position.



**Fig. 13.** A face  $\sigma \in C_{3,13}$ .

*Example 4.1.* Let  $\sigma \in C_{3,13}$  be the configuration in Figure 13. Here  $\pi(\sigma) = \pi_o(\sigma) = \{(1,1), (1,4), (1,7)\}$  and the intervals  $N(\sigma) = \{N_1, N_2, N_3\}$  are given by

$$\begin{aligned} N_1 &= \{(1,8), (1,9), (1,10), (1,11), (1,12), (1,13), (1,1)\}, \\ N_2 &= \{(1,2), (1,3), (1,4)\}, \text{ and} \\ N_3 &= \{(1,5), (1,6), (1,7)\}. \end{aligned}$$

In the interval  $N_3$ , the positions  $(1,5)$  and  $(1,7)$  are odd, while in  $N_1$ , the odd positions are  $(1,8), (1,10), (1,12)$  and  $(1,1)$ . The signature of  $\sigma$  is  $[7, 3, 3]$  which is the same as  $[3, 7, 3]$  or  $[3, 3, 7]$ .

We start by defining a closure operator  $\sigma \mapsto \widehat{\sigma}$  on the poset  $P(I(C_{m,n}))$ . The idea is to use the equivalence relation defined by  $\sigma \sim \tau$  if and only if  $\widehat{\sigma} = \widehat{\tau}$  to group faces in equivalence classes. By constructing matchings on the non-singleton classes of  $\sim$  we can then restrict our study to a particular collection of singleton classes. As before, let  $\pi(\sigma) = \{x_1, \dots, x_k\}$ . The face  $\widehat{\sigma}$  is defined as follows.

- $\sigma \subseteq \widehat{\sigma}$ .
- For each  $i = 1, \dots, k$ , if  $x_i$  is in an even position and if  $x_i$  is free in  $\sigma$ , then  $x_i \in \widehat{\sigma}$ .

Let  $\sim$  be the equivalence relation described above and let  $X_\sigma$  be the equivalence class of  $\sigma$  with respect to  $\sim$ . That is,

$$X_\sigma = \{\tau \in I(C_{m,n}) \mid \tau \sim \sigma\}.$$

Let  $\sigma$  be the face from Example 4.1. Then,  $\widehat{\sigma} = \sigma + (1, 9)$  and  $X_\sigma = \{\sigma, \sigma + (1, 9)\}$ .

**Lemma 4.1** *Let  $\sigma \in I(C_{m,n})$ .*

- Let  $x$  be a free and even position in an interval  $N_i \in N(\sigma)$ . Then,  $\sigma \sim \sigma + x$ ,  $\pi_o(\sigma + x) = \pi_o(\sigma)$ , and  $\pi_e(\sigma + x) = \pi_e(\sigma) + x$ .
- Assume that  $|N(\sigma)| > 1$ . Let  $x_i \in \pi_e(\sigma)$ . Then,  $\sigma \sim \sigma - x_i$ ,  $\pi_o(\sigma - x_i) = \pi_o(\sigma)$ , and  $\pi_e(\sigma - x_i) = \pi_e(\sigma) - x_i$ .

*Proof.* The first part follows from the observation that adding the element  $x$  to  $\sigma$  divides the interval  $N_i$  into one even interval, with last position in  $x$ , and one interval of the same parity as  $N_i$ , given by the position  $x_i$ .

The second part holds since gluing an even interval with its following interval (which is necessarily different due to the lower bound condition on  $N(\sigma)$ ) creates an interval of the same parity as the latter one. The parity of the positions in the new interval are the same as in the old intervals and the vacated position  $x_i$  is free and even in  $\sigma - x_i$ , which shows that  $\sigma \sim \sigma - x_i$ .  $\square$

We partition the faces  $\sigma \in I(C_{m,n})$  into three sets  $P_1$ ,  $P_2$  and  $P_3$ , depending on the signature and the size of  $X_\sigma$ .

- $P_1$  is the set of  $\sigma$  for which  $|X_\sigma| > 1$  and for which there is at least one odd interval in  $N(\sigma)$ .
- $P_2$  is the set of  $\sigma$  for which  $|X_\sigma| > 1$  and all the intervals in  $N(\sigma)$  are even.
- $P_3$  is the set of  $\sigma$  for which  $X_\sigma$  is a singleton.

Note that Lemma 4.1 implies that if  $\sigma \in P_i$ , then  $\tau \in P_i$  for all  $\tau \in X_\sigma$ ,  $i \in \{1, 2, 3\}$ .

We can use Lemma 4.1 to describe the sets  $X_\sigma$  more explicitly. The idea is to use the first part to move from  $\sigma$  to  $\widehat{\sigma}$ . In the process, we add only even intervals. Then, we can use the second part to move from  $\widehat{\sigma}$  to any  $\tau \in X_\sigma$  by doing the reverse operation of gluing an even interval with its following interval.

**Corollary 4.2** *Assume that  $\sigma \in P_1$ . Then,  $\tau \in X_\sigma$  if and only if  $\tau$  can be obtained from  $\widehat{\sigma}$  by removing a subset of  $\pi_e(\widehat{\sigma})$ .*

**Corollary 4.3** *Assume that  $\sigma \in P_2$ . Then,  $\tau \in X_\sigma$  if and only if  $\tau$  can be obtained from  $\widehat{\sigma}$  by removing a proper subset of  $\pi_e(\widehat{\sigma})$ .*

We can therefore describe the structure of the equivalence classes in the following way.

$$X_\sigma = \begin{cases} \{\widehat{\sigma} \setminus S \mid S \subseteq \pi_e(\widehat{\sigma})\} & \text{if } \sigma \in P_1, \\ \{\widehat{\sigma} \setminus S \mid S \not\subseteq \pi(\widehat{\sigma})\} & \text{if } \sigma \in P_2, \\ \{\sigma\} & \text{if } \sigma \in P_3. \end{cases} \quad (9)$$

Using the description of  $X_\sigma$  in (9), we can show that the equivalence classes in  $P_1$  and  $P_2$  behave well. These results are given in Lemma 4.4 and Lemma 4.5.

**Lemma 4.4** *Let  $\sigma \in P_1$ . Then,*

$$\sum_{\tau \in X_\sigma} (-1)^{|\tau|} = 0.$$

*Proof.* Since  $|X_\sigma| > 1$ , we know that  $\pi_e(\widehat{\sigma})$  is non-empty. In particular, we can choose an  $x_i \in \pi_e(\widehat{\sigma})$  and match  $\tau - x_i$  with  $\tau + x_i$  for each  $\tau \in X_\sigma$ .  $\square$

**Lemma 4.5** *Let  $\sigma \in P_2$ . Then,*

$$\sum_{\tau \in X_\sigma} (-1)^{|\tau|} = (-1)^{|\sigma \setminus \pi(\sigma)|+1}.$$

*Proof.* Note that for all  $\sigma \in P_2$ , we have  $\pi_e(\widehat{\sigma}) = \pi(\widehat{\sigma})$ . According to (9), the set  $X_\sigma$  consists of the faces  $\widehat{\sigma} \setminus S$  where  $S \not\subseteq \pi(\widehat{\sigma})$ . Therefore,

$$\sum_{\tau \in X_\sigma} (-1)^{|\tau|} = \left( \sum_{\tau=\widehat{\sigma} \setminus S, S \subseteq \pi(\widehat{\sigma})} (-1)^{|\tau|} \right) - (-1)^{|\widehat{\sigma} \setminus \pi(\widehat{\sigma})|} = 0 - (-1)^{|\sigma \setminus \pi(\sigma)|}.$$

$\square$

Lemma 4.5 needs a bit more work to use than Lemma 4.4. In order to find the sum of  $(-1)^{|\sigma|}$  over all  $\sigma \in P_2$ , we must use Lemma 4.5 on each equivalence class in  $P_2$ . Luckily, there is an easy way to describe all equivalence classes. For  $\sigma \in P_2$ , let  $Y_\sigma = \{\tau \in P_2 \mid \tau \setminus \pi(\tau) = \sigma \setminus \pi(\sigma)\}$ . That is,  $Y_\sigma$  is the set of faces in  $P_2$  which are equal to  $\sigma$  on rows 2 through  $m$ . Then, since all faces in  $P_2$  contain only even intervals,  $Y_\sigma$  can be partitioned into two sets,

$$Y_\sigma^1 = \{\tau \in Y_\sigma \mid \pi(\tau) \subseteq \{(1, 1), (1, 3), \dots, (1, n-1)\}\}$$

and

$$Y_\sigma^2 = \{\tau \in Y_\sigma \mid \pi(\tau) \subseteq \{(1, 2), (1, 4), \dots, (1, n)\}\}$$

Now, the set of non-empty  $Y_\sigma^1$  and  $Y_\sigma^2$  equals the set of equivalence classes in  $P_2$ . This observation leads to the following lemma.

**Lemma 4.6**

$$\sum_{\sigma \in P_2} (-1)^{|\sigma|} = -2 \sum_{\tau \in I(C_{m-1, n})} (-1)^{|\tau|} + \begin{cases} 2 \cdot (-1)^{mn/4} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

*Proof.* Following the previous discussion, we reorder the sum to be over the sets  $Y_\sigma$  and since  $Y_\sigma$  is uniquely determined by rows 2 through  $m$  of  $\sigma$ , we will sum over faces  $\mu \in C_{m-1,n}$ , abusing notation slightly to write  $Y_\mu = Y_\sigma$  when  $\mu$  equals  $\sigma \setminus \pi(\sigma)$  as sets.

$$\sum_{\sigma \in P_2} (-1)^{|\sigma|} = \sum_{Y_\sigma} \sum_{\tau \in Y_\sigma} (-1)^{|\tau|} = \sum_{\mu \in C_{m-1,n}} \left( \sum_{\tau \in Y_\mu^1} (-1)^{|\tau|} + \sum_{\tau \in Y_\mu^2} (-1)^{|\tau|} \right)$$

When  $Y_\mu^1$  and  $Y_\mu^2$  in the inner sum are non-empty, we can use Lemma 4.5. The set  $Y_\mu^1$  is empty precisely when the first row of  $\mu$  (which is the second row of the corresponding  $\sigma$ ) has particles in positions  $(1, 1), (1, 3), \dots, (1, n-1)$ . Let  $X_1$  be the set of such  $\mu$  and let  $X_2$  be the set of  $\mu$  which has particles in positions  $(1, 2), (1, 4), \dots, (1, n)$ . This gives us the following expression.

$$\sum_{\sigma \in P_2} (-1)^{|\sigma|} = \sum_{\mu \in C_{m-1,n}} 2 \cdot (-1)^{|\mu|+1} - \sum_{\tau \in X_1} (-1)^{|\tau|+1} - \sum_{\tau \in X_2} (-1)^{|\tau|+1}$$

The lemma follows by constructing a matching tree for the set  $X_1$  (or equivalently, for  $X_2$ ). This set can be perfectly matched when  $m$  is odd and can be matched except for a single face with  $mn/4$  particles when  $m$  is even.  $\square$

It remains to describe the faces  $\sigma$  which are in the set  $P_3$ . In order to do this, we will further divide this set into three disjoint sets,  $Q_1$ ,  $Q_2$  and  $Q_3$ .

- $Q_1$  is the set of faces  $\sigma$  with signature  $[3, 3, \dots, 3]$ .
- $Q_2$  is the set of faces  $\sigma$  for which all the intervals are of odd length, at least one of length greater than 3, and such that all even positions in the intervals are non-free.
- $Q_3$  is the set of faces  $\sigma$  with signature  $\emptyset$ .

These sets partition the remaining faces since all  $\sigma$  with both odd and even intervals are in  $P_1$  and all  $\sigma$  with only even intervals are in  $P_2$ .

#### Lemma 4.7

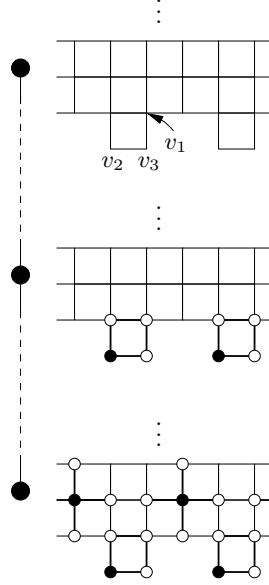
$$\sum_{\sigma \in Q_1} (-1)^{|\sigma|} = \begin{cases} 0 & \text{if } m \equiv_3 0, \\ 3 \cdot (-1)^{n/3} & \text{if } m \equiv_3 1, \\ 3 & \text{if } m \equiv_3 2. \end{cases}$$

*Proof.* Note that the faces with signature  $[3, 3, \dots, 3]$  fall into three equivalence classes of  $\sim$ . Let  $C'_{m,n} = C_{m,n} \setminus \{(1, 3i+1) \mid i = 0, \dots, n/3-1\}$ . We will first determine the homotopy type of  $I(C'_{m,n})$ . We then obtain the desired sum by determining

$$\sum_{\sigma \in C'_{m-1,n}} (-1)^{|\sigma|+n/3}$$

and multiply this by 3.

Figure 14 shows a sketch of the matching tree construction. If we split on  $v_1$ , the subtree where  $v_1$  belongs to all faces will be perfectly matched on  $v_2$ . We can therefore assume that  $v_1$  is empty, and then match on  $v_3$ . In the second graph, these vertices have been fixed. We now have two vertices of degree 1, which we



**Fig. 14.** Matching tree for faces  $\sigma$  with  $\pi(\sigma) = \{(1, 3i + 1) \mid i = 0, \dots, n/3 - 1\}$ .

can match in any order to obtain the third graph. Note that after removing the first three rows and a total of  $2n/3$  particles, the third graph is identical to the graph in the root. This provides a recursive relation. The base cases are when  $m = 1$ ,  $m = 2$  and  $m = 3$ . The first two of these leave a single configuration with  $n/3$  and  $2n/3$  particles, respectively. For  $m = 3$ , the entire complex is matched.

$$I(C'_{m,n}) \simeq \begin{cases} \bullet & \text{if } m \equiv_3 0, \\ S^{2(m+1/2)n/9-1} & \text{if } m \equiv_3 1, \\ S^{2(m+1)n/9-1} & \text{if } m \equiv_3 2. \end{cases}$$

The lemma follows.  $\square$

*Conjecture 4.1.* Let  $n$  be odd. Then, for  $\sigma \in Q_2$ ,

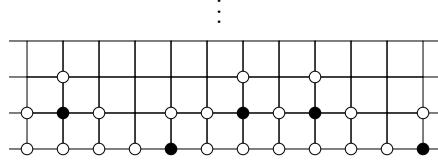
$$\sum_{\tau: \pi(\tau) = \pi(\sigma)} (-1)^{|\tau|} = 0.$$

We can prove the following, which shows a special case of Conjecture 4.1, but also holds for even  $n$ .

**Lemma 4.8** *Let  $\sigma \in Q_2$  and assume that for all intervals  $N_i \in N(\sigma)$ , it holds that  $|N_i| \geq 5$ . Then,*

$$\sum_{\tau: \pi(\tau) = \pi(\sigma)} (-1)^{|\tau|} = 0.$$

*Proof.* Figure 15 shows the fixed vertices in the lower part of a typical configuration in  $Q_2$ . In this particular example,  $n = 12$  and  $\pi(\sigma) = \{(1, 5), (1, 12)\}$ .



**Fig. 15.** Configurations from  $Q_2$  with  $n = 12$  and  $\pi(\sigma) = \{5, 12\}$ .

The same graph, when the fixed vertices are removed, is shown in the root of Figure 16. We will now construct a matching tree on this graph and show that we can match all configurations  $\tau$ , with  $\pi(\tau) = \pi(\sigma)$  perfectly.

First, we match on  $u_1$  and  $v_1$ . By doing this, we obtain two new vertices of degree 1, namely  $u_2$  and  $v_2$ . We also have the vertex  $v_3$  which had degree 1 already in the root node. We continue by matching on  $u_2, v_3$  and  $v_2$  (in any order), and obtain the final graph in Figure 16. Removing all fixed vertices in this graph yields, after a rotation of the cylinder, the graph in the root node of height 2 less. This is the recursive step. It is easy to see that the base cases, for  $m = 2$  and  $m = 3$  are perfectly matched. When  $m = 1$ , the set  $Q_2$  is empty.

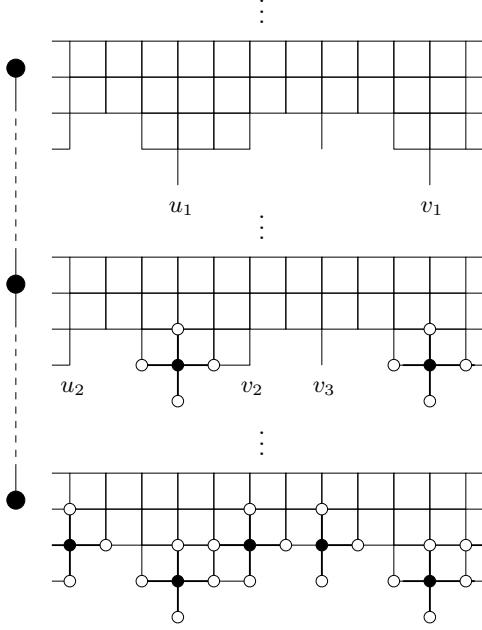
We argue that this construction is extendable to any  $\pi(\sigma)$ , where  $\sigma \in Q_2$ . Take such a  $\sigma$  and an interval  $N_i \in N(\sigma)$ . On the second row, there are  $(|N_i| - 3)/2$  forced particles, due to the fact that all even positions in  $N_i$  must be non-free. Let  $y_i$  be the last such particle, which always exists since  $|N_i| \geq 5$ . Then, for each  $i$ ,  $y_i$  and  $x_i$  forces a degree-1 vertex. In the example these degree-1 vertices are  $u_1$  and  $v_1$ . We start by matching on these vertices. In the second step, we note that for each interval, there will now be  $(|N_i| - 3)/2$  vertices of degree 1 on row 3. In the example these vertices are  $u_2$  for the interval  $\{(1, 1), \dots, (1, 5)\}$  and  $v_2$  and  $v_3$  for the interval  $\{(1, 6), \dots, (1, 12)\}$ . As in the example, when  $m = 2$  and  $m = 3$  everything can be perfectly matched. Therefore, the number of intervals does not matter, as long as they are all odd, and all of size greater than or equal to 5.  $\square$

The final set  $Q_3$  will function as an inductive step, since we have:

$$\sum_{\sigma \in Q_3} (-1)^{|\sigma|} = \sum_{\sigma \in I(C_{m-1,n})} (-1)^{|\sigma|}. \quad (10)$$

**Proposition 4.9** *Conjecture 4.1 implies Conjecture 2.1.*

*Proof.* Let  $n$  be odd. We know from Lemma 4.4 that the alternating sum over  $P_1$  is 0. Since  $m$  is odd, we further know that  $P_2$  is empty. For  $P_3$ , we use Lemma 4.7 for  $Q_1$  and Conjecture 4.1 for  $Q_2$ . Finally, as noted in (10), we get an inductive



**Fig. 16.** Matching tree recursion for the configurations from Figure 15.

step from  $Q_3$ .

$$\begin{aligned}
 Z(C_{m,n}) &= \sum_{\sigma \in P_1} (-1)^{|\sigma|} + \sum_{\sigma \in P_2} (-1)^{|\sigma|} + \sum_{\sigma \in P_3} (-1)^{|\sigma|} = \\
 &= 0 + 0 + \sum_{\sigma \in Q_1} (-1)^{|\sigma|} + \sum_{\sigma \in Q_2} (-1)^{|\sigma|} + \sum_{\sigma \in Q_3} (-1)^{|\sigma|} = \\
 &= Z(C_{m-1,n}) + \begin{cases} 0 & \text{if } m \equiv_3 0 \text{ and } 3|n, \\ -3 & \text{if } m \equiv_3 1 \text{ and } 3|n, \\ 3 & \text{if } m \equiv_3 2 \text{ and } 3|n. \end{cases}
 \end{aligned}$$

The base case is  $Z(C_{1,n})$  which is the alternating sum of the independence complex on an  $n$ -cycle. Its values are given by the following expression.

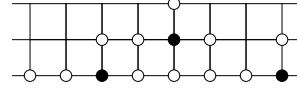
$$Z(C_{1,n}) = \begin{cases} -2 & \text{if } n \equiv_3 0, \\ (-1)^{(n-1)/3} & \text{if } n \equiv_3 1, \\ (-1)^{(n+1)/3} & \text{if } n \equiv_3 2. \end{cases}$$

Since  $n$  is odd, this can be simplified to  $Z(C_{1,n}) = -2$  if  $3|n$  and  $Z(C_{1,n}) = 1$  otherwise. Together with the inductive step, this shows the proposition.  $\square$

When  $n = 3$ , the set  $Q_2$  is empty and for  $n = 5, 7$  and  $9$ , Lemma 4.8 covers all the cases of Conjecture 4.1. Therefore, Conjecture 2.1 holds for these values of  $n$ .

Conjecture 4.1 does not hold for all even  $n$ . A counterexample is given in Figure 17, where  $m = 3, n = 8$  and  $\pi(\sigma) = \{(1,3), (1,8)\}$ . A straightforward application of the matching tree construction produces a matching on the faces  $\tau$  with

$\pi(\tau) = \pi(\sigma)$  which leaves a single face with six particles. Thus,  $\sum_{\tau: \pi(\tau)=\{3,8\}} (-1)^{|\tau|} = 1$  in this case.



**Fig. 17.** Counter-example to Conjecture 4.1 when  $n$  is even.

Still, we can determine  $Z(C_{m,n})$  for some small, even  $n$ . When  $n = 6$ , both  $P_1$  and  $Q_2$  are empty. Lemma 4.6, Lemma 4.7 and (10) gives

$$\sum_{\sigma \in I(C_{6,n})} (-1)^{|\sigma|} = - \sum_{\sigma \in I(C_{m-1,n})} (-1)^{|\sigma|} + \begin{cases} 0 + 2 \cdot (-1)^{m/2} & m \equiv_6 0 \\ 3 + 0 & m \equiv_6 1 \\ 3 + 2 \cdot (-1)^{m/2} & m \equiv_6 2 \\ 0 + 0 & m \equiv_6 3 \\ 3 + 2 \cdot (-1)^{m/2} & m \equiv_6 4 \\ 3 + 0 & m \equiv_6 5 \end{cases}$$

Therefore,  $Z(C_{m,6})$  is the following periodic sequence, starting from  $m = 1$ . The period is 12.

$$2, -1, 1, 4, -1, -1, 4, 1, -1, 2, 1, 1, 2, -1, 1, \dots$$

		m									
		1	2	3	4	5	6	7	8	9	10
n	2										
	4										
	6										
	8		8		8		8	-8	8	-8	8
	10				10				-10		
	12					12					
	14					14			14		14
	16		8		8		8	24	8	-8	24
	18					18			36		
	20				10				50		
	22						22			88	
	24		8		8		20	-8	8	64	8

**Table 8.** Alternating sums over  $Q_2$  for small  $m$  and  $n$ .

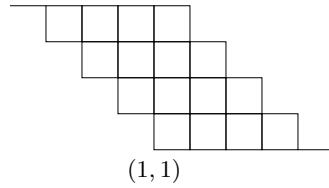
In Table 8 we have taken the values of  $Z(C_{m,n})$  from Table 3 and removed the part which can be explained using Lemma 4.6, Lemma 4.7 and (10) similarly to what we did in the previous example. What remains in the table is therefore the alternating sums over the faces in  $Q_2$ , or alternatively over those faces in  $Q_2$

which are not matched by Lemma 4.8. These are still unexplained, but several interesting patterns seem to emerge.

## 5 Transfer matrices

In [3], Bousquet-Mélou, Linusson and Nevo determined the partition function of the hard particle model on a parallelogram shaped region  $P_{m,n}$  of the square grid. We describe it by letting  $(a, b) \in V(P_{m,n})$  if

$$1 \leq a \leq m \quad \text{and} \quad 1 - a < b \leq 1 - a + n.$$



**Fig. 18.** The graph  $P_{5,6}$ .

Figure 18 shows the region  $P_{5,6}$ . The skewed border allowed them to find a Morse-matching for the complex  $I(P_{m,n})$  and determine its homotopy type. They further realised that this could be used to produce parts of the spectrum of the transfer matrix for the cylinder  $C_{m,n}$ . In this section, we will review this idea and show how we can complete the case for  $m = 4$ . We also find the missing eigenvalue when  $m = 6$ . Unfortunately, we were not able to generalise our findings to new, unknown spectra.

We start by defining the two transfer matrices of interest. For a fixed  $m$ , the matrix  $T(m)$  describes the configurations on the region  $P_{m,n}$  and  $T'(m)$  describes the configurations on the region  $S_{m,n}$ . Throughout this section,  $i$  will denote the square root of  $-1$ .

Let  $T(m)$  be the  $2^m \times 2^m$ -matrix, indexed by subsets of  $[m]$  on both rows and columns, and defined as follows.

$$T(m)_{A,B} = \begin{cases} i^{|A|+|B|} & \text{if } A \cap B = A \cap B' = \emptyset, \\ 0 & \text{otherwise,} \end{cases} \quad (11)$$

where  $B' = \{j+1 \mid j \in B \cap [m-1]\}$ .

Let  $T'(m)$  be the  $F_{m+1} \times F_{m+1}$ -matrix, indexed by independent sets on a path with  $m$  vertices, and defined as follows.

$$T'(m)_{A,B} = \begin{cases} i^{|A|+|B|} & \text{if } A \cap B = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

We here follow [3] and in contrast to the example in Section 2.4 distribute the weights of the entries in row  $A$ , column  $B$  and row  $B$ , column  $A$  evenly.

As was illustrated in Section 2.4, we can determine the values of  $Z(C_{m,n})$  by taking the trace of the  $n$ th power of either one of these matrices. Therefore, the following relation between the generating functions of  $Z(C_{m,n})$  holds.

$$\mathrm{tr}(1-tT(m))^{-1} - 2^m = \sum_{n \geq 1} Z(P_{m,n})t^n = \mathrm{tr}(1-tT'(m))^{-1} - F_{m+1} \quad (13)$$

We conclude that the spectra of  $T(m)$  and  $T'(m)$  are equal, apart from the multiplicity of the null eigenvalue due to the difference in size.

Let  $A, B \subseteq [m]$  and assume that we prescribe the particles given by  $A$  and  $B$  on the left and right boundary of  $P_{m,n}$ . That is, when  $j \in A$ , then the  $j$ th particle from the top on the left boundary is present, and otherwise it is missing in each configuration and similarly for  $B$  on the right boundary. This induces a subgraph of  $P_{m,n}$  which we will denote by  $P_{m,n}(A, B)$ .

Let  $G_{A,B}(t)$  denote the element is row  $A$ , column  $B$  of  $(1-tT(m))^{-1}$ . Then,  $G_{A,B}(t)$  is the generating function for configurations on  $P_{m,n}(A, B)$ , weighted as given by (11) and has the following expression.

$$G_{A,B}(t) = \delta_{A,B} + (-i)^{|A|+|B|} \sum_{n \geq 1} Z(P_{m,n+1}(A, B))t^n \quad (14)$$

We can then split the left-hand side of (13) into

$$\mathrm{tr}(1-tT(m))^{-1} = \sum_{A \subseteq [m]} G_{A,A}(t). \quad (15)$$

The observation made in [3] was that parts of the spectra can be recovered by finding  $G_{A,A}(t)$  for some  $A$  and that this can be done in the model for  $P_{m,n}$  which is more easily determined than that of  $C_{m,n}$ .

The characteristic polynomial of  $T'(4)$  is determined both in [3] and [9] and is given by  $(1-t+t^2)(1-t^2)(1-t^4)$ . Let  $G_{A,B}^{(N)}(t)$  denote the first  $N$  terms in the expansion of (14). By finding a Morse-matching for the region in Figure 18, Bousquet-Mélou, Linusson and Nevo effectively calculated  $G_{\emptyset,\emptyset}(t)$ . When  $m = 4$ , they found that

$$G_{\emptyset,\emptyset}(t) = \frac{1}{1-t+t^2}. \quad (16)$$

This determined

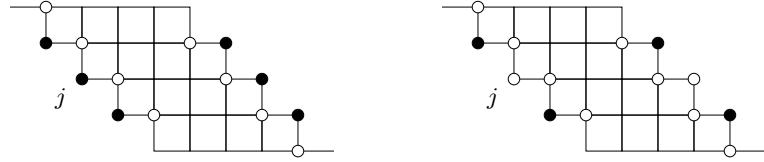
This recovers one of the factors of  $T'(4)$ . Using this case, and a recursion for  $Z(P_{m,n}(\{2,3\}, \{2,3\}))$ , they could then determine the function  $G_{\{2,3\}, \{2,3\}}(t)$ .

$$G_{\{2,3\}, \{2,3\}}(t) = \frac{1+t^2+t^3}{(1+t^3)(1-t^4)} \quad (17)$$

We will now determine the remaining cases for  $m = 4$ . We start with the following lemma.

**Lemma 5.1** *Assume that  $j-1, j, j+1 \in A$  for some  $j$  and let  $B = A \setminus \{j\}$ . Then,*

$$G_{A,A}(t) + G_{B,B}(t) = 2.$$



**Fig. 19.** Two regions differing only in the position  $j$ .

*Proof.* Figure 19 illustrates that the regions remaining after fixing the boundaries to  $A$  and  $B$  respectively are equal. Since the addition of two elements does not change the parity, we have  $Z(P_{m,n}(A, A)) = Z(P_{m,n}(B, B))$ . Since furthermore  $(-i)^{|A|+|A|} = -(-i)^{|B|+|B|}$ , all terms except the constants cancel and we have  $G_{A,A}(t) + G_{B,B}(t) = \delta_{A,A} + \delta_{B,B} = 2$ .  $\square$

Using Lemma 5.1 we can avoid calculating certain sets by choosing a partial matching on  $2^{[4]}$  which satisfies the condition in the lemma. We will in this way match  $\{1, 2, 3\}$  with  $\{1, 3\}$ ,  $\{2, 3, 4\}$  with  $\{2, 4\}$  and  $\{1, 2, 3, 4\}$  with  $\{1, 3, 4\}$ . This leaves only  $\{1, 2, 4\}$  unmatched among the potential candidates for Lemma 5.1. We can produce its generating function by noting that  $G_{[4],[4]}(t) = 1 + t^2 G_{\emptyset,\emptyset}(t)$ . Lemma 5.1 applied to  $A = [4]$  and  $i = 3$  gives us

$$G_{\{1,2,4\},\{1,2,4\}}(t) = 1 - t^2 G_{\emptyset,\emptyset}(t) = \frac{1-t}{1-t+t^2}. \quad (18)$$

The sets that remain are the singleton sets,  $\{1, 2\}$ ,  $\{3, 4\}$  and  $\{1, 4\}$ . By symmetry, we can restrict our calculations to the sets  $\{3\}$ ,  $\{4\}$ ,  $\{1, 2\}$  and  $\{1, 4\}$ . We present the calculations for  $\{4\}$  and  $\{1, 4\}$ .

We start with the case  $A = \emptyset$  and  $B = \beta$ , where  $\beta$  is an arbitrary subset of  $[m]$ . When  $j \geq 4$ , we have the recursive relation  $Z(P_{4,j}(\emptyset, \beta)) = -Z(P_{4,j-4}(\emptyset, \beta))$ . Thus,  $G_{\emptyset,\beta}(t) = G_{\emptyset,\beta}^{(4)}(t) - t^3 G_{\emptyset,\beta}(t)$  which gives us the following general expression.

$$G_{\emptyset,\beta}(t) = \frac{G_{\emptyset,\beta}^{(4)}(t)}{1+t^3} \quad (19)$$

The recursive relation which was used to produce (19), and those that follow, are obtained using simple matching trees. We present some, but not all trees here. Figure 20 illustrates a matching tree when  $A = \{4\}$ . This gives us the following recursive relation for the case  $A = B = \{4\}$ .

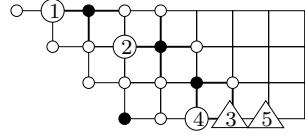
$$Z(P_{4,j}(\{4\}, \{4\})) = Z(P_{4,j-4}(\{4\}, \{4\})) + Z(P_{4,j-4}(\emptyset, \{4\})) \quad (20)$$

For the second term, we calculate  $G_{\emptyset,\{4\}}^{(4)}(t) = -it$ , which, using (19), gives  $G_{\emptyset,\{4\}}(t) = -it/(1+t^3)$ . Plugging this term into (20) gives the expression

$$G_{\{4\},\{4\}}(t) = G_{\{4\},\{4\}}^{(5)}(t) + t^4 (G_{\{4\},\{4\}}(t) - 1) + it^4 G_{\emptyset,\{4\}}(t), \quad (21)$$

which, after calculating  $G_{\{4\},\{4\}}^{(5)}(t) = 1+t^4$ , gives the final function for  $A = B = \{4\}$ , and by symmetry for  $A = B = \{1\}$ .

$$G_{\{4\},\{4\}}(t) = G_{\{1\},\{1\}}(t) = \frac{1+t^3+t^5}{(1+t^3)(1-t^4)} \quad (22)$$



**Fig. 20.** The large circles indicate match nodes and the large triangle indicate split nodes. The numbers give the order in which the nodes are matched or split. At 5, the two branches of the recursion are obtained.

For  $A = B = \{1, 4\}$ , we have the recursion

$$Z(P_{4,j}(A, A)) = Z(P_{4,j-1}(\emptyset, A)) + Z(P_{4,j-4}(A, A)). \quad (23)$$

We proceed by summing over  $j$  and calculating the base cases.

$$\begin{aligned} G_{\{1,4\}, \{1,4\}}(t) \\ = G_{\{1,4\}, \{1,4\}}^{(5)}(t) - t \left( G_{\emptyset, \{1,4\}}(t) - G_{\emptyset, \{1,4\}}^{(4)}(t) \right) + t^4 (G_{\{1,4\}, \{1,4\}}(t) - 1) \\ = \left( 1 + t^2 + t^4 + \frac{t^2}{1+t^3} - t^2 - t^4 \right) / (1-t^4) \end{aligned} \quad (24)$$

In the last equality, we have calculated the polynomials

$$G_{\{1,4\}, \{1,4\}}^{(5)}(t) = 1 + t^2 + t^4 \text{ and } G_{\emptyset, \{1,4\}}^{(4)}(t) = -t,$$

and used the expression for  $G_{\emptyset, \{1,4\}}(t)$  in (19). It turns out that  $G_{\{1,4\}, \{1,4\}}(t) = G_{\{2,3\}, \{2,3\}}(t)$ .

The final two cases, when  $A = B = \{3\}$  and  $A = B = \{1, 2\}$ , are determined similarly, and we present them without any explicit calculations.

$$G_{\{2\}, \{2\}}(t) = G_{\{3\}, \{3\}}(t) = \frac{1 + t^3 + t^5}{(1+t^3)(1-t^4)} \quad (25)$$

Note that (25) implies that all the singleton sets have the same generating function.

$$G_{\{1,2\}, \{1,2\}}(t) = G_{\{3,4\}, \{3,4\}}(t) = 1 - \frac{t^5}{(1+t^3)(1-t^4)}. \quad (26)$$

We note that while all factors of the characteristic polynomial show up in the individual terms, we can not deduce their multiplicities until we have taken the sum over all sets. Expanding this sum into partial fractions, we arrive at the following expression.

$$\begin{aligned} \sum_{A \subseteq [4]} G_{A,A}(t) &= G_{\emptyset, \emptyset}(t) + G_{\{1,2,4\}, \{1,2,4\}}(t) + 6 + \\ &\quad + 4G_{\{4\}, \{4\}}(t) + 2G_{\{2,3\}, \{2,3\}}(t) + 2G_{\{1,2\}, \{1,2\}}(t) \\ &= 8 + \frac{1}{1-it} + \frac{1}{1+it} + \frac{2}{1+t} + \frac{2}{1-t} + \frac{1}{1-\alpha t} + \frac{1}{1-\bar{\alpha}t}, \end{aligned}$$

where  $\alpha = (1 + i\sqrt{3})/2$  and  $\bar{\alpha}$  are the two roots of  $1 - t + t^2$ . As expected, we recover the roots and multiplicities of the characteristic polynomial of  $T(4)$ , which apart from the 8 zeroes equals the characteristic polynomial of  $T'(4)$ .

Next, we determine the factor of the characteristic polynomial for  $m = 6$  which does not follow from the case  $A = B = \emptyset$ . The characteristic polynomial for  $m = 6$  is given by  $(1 - t^4)^2(1 - t^{14})/(1 + t)$  [3,9] and the function  $G_{\emptyset, \emptyset}(t)$  follows from the general result in [3].

$$G_{\emptyset, \emptyset}(t) = \frac{1}{1 - t^{14}} \left( 1 + t + t^3 \frac{1 - t^{12}}{1 - t^3} \right) \quad (27)$$

This leaves the factor  $1 - t^4$  unexplained. We show that this factor appears in the generating function  $G_{A, A}(t)$  for  $A = \{2, 3, 6\}$ .

As in the previous case, we start by deriving an expression for  $G_{\emptyset, \beta}(t)$  for an arbitrary  $\beta$ . The matching leading up to the recursion contains more steps, but is straightforward using matching trees. The final formula is

$$G_{\emptyset, \beta}(t) = \frac{G_{\emptyset, \beta}^{(15)}(t)}{1 - t^{14}}. \quad (28)$$

We will also make use of the recursive relation

$$Z(P_{6,j}(\{2\}, \{2, 3, 6\})) = -Z(P_{6,j-6}(\emptyset, \{2, 3, 6\})), \quad (29)$$

from which we derive that

$$\begin{aligned} G_{\{2\}, \{2, 3, 6\}}(t) &= G_{\{2\}, \{2, 3, 6\}}^{(7)}(t) + it^6 G_{\emptyset, \{2, 3, 6\}}(t) \\ &= t^4 + it^6 \left( \frac{-i(t + t^4 + t^{12})}{1 - t^{14}} \right) = \frac{t^4 (1 + t^3 + t^6)}{1 - t^{14}}. \end{aligned} \quad (30)$$

In the last equality we have used that  $G_{\emptyset, \{2, 3, 6\}}^{(15)}(t) = -i(t + t^4 + t^{12})$ . The main recursion for  $A = \{2, 3, 6\}$  is given by the following relation.

$$Z(P_{6,j}(A, A)) = Z(P_{6,j-4}(A, A)) - Z(P_{6,j-1}(\emptyset, A)) - Z(P_{6,j-4}(\{2\}, A)) \quad (31)$$

The matching tree needed to produce this relation is shown in Figure 21. The terms  $Z(P_{6,j-1}(\emptyset, A))$  and  $Z(P_{6,j-4}(\{2\}, A))$  in the right-hand side of (31) are obtained in nodes 1 and 4, respectively. After summation over  $i$  we have

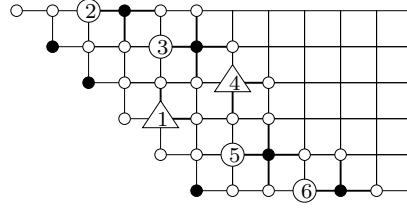
$$\begin{aligned} G_{\{2, 3, 6\}, \{2, 3, 6\}}(t) &= G_{\{2, 3, 6\}, \{2, 3, 6\}}^{(5)}(t) + t^4 (G_{\{2, 3, 6\}, \{2, 3, 6\}}(t) - 1) \\ &\quad - it \left( G_{\emptyset, \{2, 3, 6\}}(t) - G_{\emptyset, \{2, 3, 6\}}^{(4)}(t) \right) + t^4 G_{\{2\}, \{2, 3, 6\}}(t) \end{aligned} \quad (32)$$

Calculations of initial terms in the relevant series provides the polynomials

$$G_{\{2, 3, 6\}, \{2, 3, 6\}}^{(5)}(t) = 1 - t^2 + t^4 \text{ and } G_{\emptyset, \{2, 3, 6\}}^{(4)}(t) = -it,$$

so (32) evaluates to

$$G_{\{2, 3, 6\}, \{2, 3, 6\}}(t) = \frac{(1 + t - t^5 - t^6 - t^7 + t^{11} + t^{12})(1 - t)}{(1 - t^{14})(1 - t^4)}. \quad (33)$$



**Fig. 21.** A matching for  $P_{6,j}(\{2,3,6\}, \{2,3,6\})$ .

This recovers the missing factor  $1 - t^4$ .

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